FIRST ORDER DIFFERENTIAL EQUATIONS

A differential equation is an equation involving one or more derivatives.

Example

1st Order

1) $\left(x^2-1\right)\frac{dy}{dx}+2xy=x$

First Order Differential Equation

2nd Order

2) $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+4y=0$

Second Order Differential Equation

VERIFICATION OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

1) Verify that the function $y=Ae^{5x}+Be^{-2x}$,where A and B are constants, satisfies a particular differential equation.

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}-10y=0$$

Worksheet – Verification of Differential Equations

Advanced Higher Verification of Differential Equation 1 *Unit 1 Outcome 1.4*

1. Verify that the function $y = x^4$ satisfies the differential equation.

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$$

2. Verify that the function $y = Ae^{3x} + Be^{x}$ where A and B are constants, satisfies the differential equation.

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

3. Verify that the function $y = Ae^{x} + Be^{2x}$ where A and B are constants, satisfies the differential equation.

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

4.a) Verify that the function $y = A\sin 2x + B\cos 2x$ where A and B are constants, satisfies the differential equation.

$$\frac{d^2y}{dx^2} + 4y = 0$$

b) Verify that, in general, the function $y = A\sin kx + B\cos kx$ where A, B and k are constants, satisfies the differential equation.

$$\frac{d^2y}{dx^2} + k^2y = 0$$

5.a) Verify that the function $y = Ae^{3x} + Be^{-3x}$ where A and B are constants, satisfies the differential equation.

$$\frac{d^2y}{dx^2} - 9y = 0$$

b) Verify that, in general, the function $y = Ae^{kx} + Be^{-kx}$ where A, B and k are constants, satisfies the differential equation.

$$\frac{d^2y}{dx^2} - k^2y = 0$$

6. Verify that the function $y = x^n$, where n is a constant, satisfies the differential equation.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - n^2y = 0$$

7. Verify that the function $y = Ae^{5x} + Be^{-3x} - \frac{1}{7}e^{4x}$, where A and B are constants, satisfies the differential equation.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 15y = e^{4x}$$

8. Verify that the function $y = e^{x}\sin x$ satisfies the differential equation.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

9. Verify that the function $y = x^3\ln x$ satisfies the differential equation.

$$x^2\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 9y = 0$$

10. Verify that the function $y = e^{-x}\cos 2x$ satisfies the differential equation.

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$$

GENERAL SOLUTIONS OF DIFFERENTIAL EQUATIONS

A linear first order differential equation $\left(4-x^2\right)\frac{dy}{dx}+3y=\left(4-x^2\right)^2$ has only $\frac{dy}{dx}$ and y.

$\left[\text{Not } y^2,\ y\frac{dy}{dx},\ y^3,\ \cos y \text{ etc.}\right]$

Differential equations are usually solved by integration First Order – single integration. Contains a **constant**.

A solution containing a constant of integration is called a **GENERAL SOLUTION**.

The constant can be any value therefore there is an infinite number of solutions.

1) The gradient of the tangent to a curve is given by the d.e. $\frac{dy}{dx}=2x-4$.
Find the equation of the curve.

If we are given an INITIAL CONDITION, eg we know that point (1,0) lies on the curve, we can obtain the **PARTICULAR SOLUTION**.

Scholar p63 Q50 - 53

FIRST ORDER DIFFERENTIAL EQUATIONS WITH SEPARABLE VARIABLES

One way to solve 1st Order D.E is to separate variables in the differential equation.

$f(y)dy = g(x)dx$ $\qquad$ $f(y)$ – function of only y $\qquad$ $g(x)$ – function of only x

Integrate both sides with respect to the appropriate variable.

$$\boxed{\int f(y)dy = \int g(x)dx}$$

Examples

1) Find the general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y}$.

2) Find the general solution of the differential equation $e^{4y}\dfrac{dy}{dx} - x = 2$.

3) Find the general solution of the differential equation $\frac{dy}{dx} = \frac{2}{\sin y}$.

4) Find the general solution of the differential equation $\frac{dy}{dx} = 2x\sqrt{9 - y^2}$.

Worksheet – General Solutions of Differential Equations 1 **Scholar p65 Q54 - 56**

Advanced Higher General - Differential Equation 1 *Unit 1 Outcome 1.4*

1. Find the general solution of each differential equation below:

a) $\frac{dy}{dx} = \frac{x}{2y}$ b) $\frac{dy}{dx} = \frac{x}{y^2}$ c) $\frac{dy}{dx} = \frac{x^3}{y}$

d) $\frac{dy}{dx} = \frac{x+2}{y}$ e) $2y\frac{dy}{dx} = 3x^2$ f) $y^2\frac{dy}{dx} = x^5$

g) $\frac{dy}{dx} = \frac{2}{\cos y}$ h) $\frac{dy}{dx} = \frac{\cos x}{y}$ i) $\sin y\frac{dy}{dx} = 1$

j) $\cos 2y\frac{dy}{dx} - x = 0$ k) $\frac{dy}{dx} = \frac{4x}{e^y}$ l) $\frac{dy}{dx} = \frac{e^x}{y}$

m) $e^y\frac{dy}{dx} = 1$ n) $\frac{dy}{dx} = \frac{x}{e^{2y}}$ o) $3y\frac{dy}{dx} = 5x^2$

p) $x + 4y\frac{dy}{dx} = 0$ q) $y^2\frac{dy}{dx} = x + 1$ r) $\frac{dy}{dx} = \frac{\sin x}{\cos y}$

s) $\frac{dy}{dx} = \frac{x^2 - 1}{y^3}$ t) $x^2 + 3y\frac{dy}{dx} = 0$ u) $\frac{dy}{dx} = 4x\sqrt{y}$

v) $e^{4y}\frac{dy}{dx} = x(x^2+1)$ w) $y\frac{dy}{dx} = \frac{1}{\sqrt{x}}$ x) $1 + y\frac{dy}{dx} = x$

y) $y\frac{dy}{dx} = \sqrt{x}$ z) $\sin y\frac{dy}{dx} = e^{2x}$

2. Find the general solution of each differential equation below.

a) $\frac{dy}{dx} = 4x\sqrt{1-y^2}$ b) $\frac{dy}{dx} = 2x(1+y^2)$ c) $\frac{dy}{dx} = 2x\sqrt{4-y^2}$

d) $y\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ e) $(1+x^2)y^2\frac{dy}{dx} = 1$ f) $\frac{dy}{dx} = x(4+y^2)$

3.a) Use integration by parts to find $\int xe^x dx$.

b) Hence find the general solution of the differential equation $\frac{dy}{dx} = \frac{xe^x}{y}$.

4.a) Use integration by parts to find $\int x\sin x dx$.

b) Hence find the general solution of the differential equation $\frac{dy}{dx} = \frac{x\sin x}{e^y}$.

ANSWERS

1.

a) $2y^2 = x^2 + C$ b) $2y^3 = 3x^2 + C$ c) $2y^2 = x^4 + C$

d) $y^2 = x^2 + 4x + C$ e) $y^2 = x^3 + C$ f) $2y^3 = x^6 + C$

g) $\sin y = 2x + C$ h) $y^2 = 2\sin x + C$ i) $\cos y = C - x$

j) $\sin 2y = x^2 + C$ k) $y = \ln\left(2x^2 + C\right)$ l) $y^2 = 2e^x + C$

m) $y = \ln\left(x + C\right)$ n) $y = \frac{1}{2}\ln\left(x^2 + C\right)$ o) $9y^2 = 10x^3 + C$

p) $4y^2 = C - x^2$ q) $2y^3 = 3x^2 + 6x + C$ r) $\sin y = C - \cos x$

s) $3y^4 = 4x^3 - 12x + C$ t) $9y^2 = C - 2x^3$ u) $y = \left(x^2 + C\right)^2$

v) $y = \frac{1}{4}\ln\left(x^4 + 2x^2 + C\right)$ w) $y^2 = 4\sqrt{x} + C$ x) $y^2 = x^2 - 2x + C$

y) $3y^2 = 4x^{\frac{3}{2}} + C$ z) $2\cos y = C - e^{2x}$

2.

a) $y = \sin\left(2x^2 + C\right)$ b) $y = \tan\left(x^2 + C\right)$ c) $y = 2\sin\left(x^2 + C\right)$

d) $y^2 = 2\sin^{-1} x + C$ e) $y^3 = 3\tan^{-1} x + C$ f) $y = 2\tan\left(x^2 + C\right)$

3.a) $\int xe^x dx = xe^x - e^x + C$

b) $y^2 = 2\left(x - 1\right)e^x + C$

4.a) $\int x\sin x dx = -x\cos x + \sin x + C$

b) $y = \ln(\sin x - x\cos x + C)$

1) $e^{a+b} = e^a . e^b$
2) $\ln a + \ln b = \ln(ab)$
3) $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$
4) $n \ln a = \ln\left(a^n\right)$

Remember these rules to simplify general solutions.

1) Find the general solution of the differential equation $\frac{dy}{dx} = 4y$

2) Find the general solution of the differential equation $\frac{dy}{dx} = \frac{y+2}{x+1}$.

3) Find the general solution of the differential equation $\frac{dy}{dx} = \frac{y}{2x-1}$.

PARTIAL FRACTIONS AND GENERAL SOLUTION

1) a) Express $\dfrac{2x+3}{x(x+1)}$ as partial fractions.

b) Hence find the general solution of the differential equation $x(x+1)\dfrac{dy}{dx} = y(2x+3)$ expressing *y* explicitly in terms of *x*.

PARTIAL FRACTIONS AND GENERAL SOLUTION

2) a) Express $\dfrac{x+1}{x(2x+1)}$ as partial fractions.

b) Hence find the general solution of the differential equation $\dfrac{dy}{dx}=\dfrac{y(x+1)}{x(2x+1)}$ expressing *y* explicitly in terms of *x*.

Worksheet – General Solutions of Differential Equations 2

1. Find the general solution of each differential equation, in case expressing y **explicitly** in terms of x.

a) $\dfrac{dy}{dx}=y$ b) $\dfrac{dy}{dx}=2y$ c) $\dfrac{dy}{dx}=\dfrac{y}{x}$ d) $\dfrac{dy}{dx}=\dfrac{y}{x+1}$

e) $\dfrac{dy}{dx}=\dfrac{y+1}{x+2}$ f) $(x-3)\dfrac{dy}{dx}=y$ g) $\dfrac{dy}{dx}=6xy$ h) $\dfrac{dy}{dx}=3x^2(y+1)$

i) $\dfrac{dy}{dx}=\dfrac{1+y}{1+x}$ j) $\dfrac{dy}{dx}=y+1$ k) $\dfrac{dy}{dx}=\dfrac{y+2}{x}$ l) $\dfrac{dy}{dx}=\dfrac{y}{2x+1}$

m) $\dfrac{dy}{dx}=\dfrac{y+1}{x}$ n) $\dfrac{dy}{dx}=\dfrac{y+3}{x+2}$ o) $\dfrac{dy}{dx}=\dfrac{y-1}{2x-1}$ p) $\dfrac{dy}{dx}=x(y+1)$

q) $\dfrac{dy}{dx}=-y$ r) $\dfrac{dy}{dx}=\dfrac{y}{2x}$ s) $\dfrac{dy}{dx}=\dfrac{y+1}{3x}$

2. a) Express $\dfrac{2x+1}{x(x+1)}$ in partial fractions.

b) Hence find the general solution of the differential equation $x(x+1)\dfrac{dy}{dx}=y(2x+1)$, expressing y explicitly in terms of x.

3. a) Express $\dfrac{1}{x(x+1)}$ in partial fractions.

b) Hence find the general solution of the differential equation $\dfrac{dy}{dx}=\dfrac{y}{x(x+1)}$, expressing y explicitly in terms of x.

4. a) Express $\dfrac{4x+3}{x(x+1)}$ in partial fractions.

b) Hence find the general solution of the differential equation $x(x+1)\dfrac{dy}{dx}=y(3x+4)$, expressing y explicitly in terms of x.

5. a) Express $\dfrac{2x}{x^2-1}$ in partial fractions.

b) Hence find the general solution of the differential equation $\dfrac{dy}{dx}=\dfrac{2xy}{x^2-1}$, expressing y explicitly in terms of x.

6. a) Express $\dfrac{1}{x(x-1)}$ in partial fractions.

b) Hence find the general solution of the differential equation $\dfrac{dy}{dx}=\dfrac{y}{x(x-1)}$, expressing y explicitly in terms of x.

7. a) Express $\dfrac{x}{(x+2)(x+1)}$ in partial fractions.

b) Hence find the general solution of the differential equation $\dfrac{dy}{dx}=\dfrac{xy}{(x+2)(x+1)}$, expressing y explicitly in terms of x.

ANSWERS

1.

a) $y = Ae^{x}$

b) $y = Ae^{2x}$

c) $y = Ax$

d) $y = A(x+1)$

e) $y = A(x+2)-1$

f) $y = A(x-3)$

g) $y = Ae^{3x^2}$

h) $y = Ae^{x^3} - 1$

i) $y = A(1+x)-1$

j) $y = Ae^{x} - 1$

k) $y = Ax - 2$

l) $y = A\sqrt{2x+1}$

m) $y = Ax - 1$

n) $y = A(x+2)-3$

o) $y = \sqrt{2x-1}+1$

p) $y = Ae^{\frac{1}{2}x^2} - 1$

q) $y = Ae^{-x}$

r) $y = A\sqrt{x}$

s) $y = Ax^{\frac{1}{3}} - 1$

2. a) $\dfrac{2x+1}{x(x+1)} = \dfrac{1}{x} + \dfrac{1}{x+1}$ b) $y = Ax(x+1)$

3. a) $\dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{x+1}$ b) $y = \dfrac{Ax}{(x+1)}$

4. a) $\dfrac{4x+3}{x(x+1)} = \dfrac{3}{x} + \dfrac{1}{x+1}$ b) $y = Ax^3(x+1)$

5. a) $\dfrac{2x}{x^2-1} = \dfrac{1}{x-1} + \dfrac{1}{x+1}$ b) $y = A(x-1)(x+1)\left[= A(x^2-1)\right]$

6. a) $\dfrac{1}{x(x-1)} = -\dfrac{1}{x} + \dfrac{1}{x-1}$ b) $y = \dfrac{A(x-1)}{x}$

7. a) $\dfrac{x}{(x+2)(x+1)} = \dfrac{2}{x+2} - \dfrac{1}{x+1}$ b) $y = \dfrac{A(x+2)^2}{x+1}$

PARTICULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS

1) Find the particular solution of the d.e. $y^2\dfrac{dy}{dx} = 4x^2 + 1$ given that y = 2 when x = 1, expressing y explicitly in terms of x.

2) Find the particular solution of the d.e. $\dfrac{dy}{dx} = 2xy^2$ given that $y = -\dfrac{3}{10}$ when x = -2, expressing y explicitly in terms of x.

3) Find the particular solution of the d.e. $(x+4)\dfrac{dy}{dx}+1=y$ given that y = 7 when x = -1, expressing y explicitly in terms of x.

4) a) Express $\dfrac{2}{(y+1)(y+3)}$ in partial fractions.

4 b) Hence find the particular solution of the differential equation $2x\dfrac{dy}{dx} = (y+1)(y+3)$

given that $y = -\dfrac{5}{3}$ when x = -1, expressing y explicitly in terms of x.

Worksheet – Particular Solutions of Differential Equations 1 & 2

1. Find the particular solution of each differential equation, in case expressing y **explicitly** in terms of x.

a) $3y^2\dfrac{dy}{dx}=2x+1$ $\quad y=2$ when $x=2$

b) $y^2\dfrac{dy}{dx}=2x^2+1$ $\quad y=1$ when $x=1$

c) $y^4\dfrac{dy}{dx}=3x^2$ $\quad y=-1$ when $x=1$

d) $e^y\dfrac{dy}{dx}+\sin x=0$ $\quad y=0$ when $x=\dfrac{\pi}{2}$

e) $\dfrac{dy}{dx}=y$ $\quad y=2$ when $x=0$

f) $\dfrac{dy}{dx}=4xy$ $\quad y=4$ when $x=0$

g) $x\dfrac{dy}{dx}=y+2$ $\quad y=7$ when $x=3$

h) $(x+1)\dfrac{dy}{dx}=y$ $\quad y=12$ when $x=2$

i) $(x+2)\dfrac{dy}{dx}=y+1$ $\quad y=-2$ when $x=-3$

j) $\dfrac{dy}{dx}=6x(y+1)$ $\quad y=3$ when $x=0$

k) $(x+4)\dfrac{dy}{dx}+3=y$ $\quad y=13$ when $x=1$

l) $\dfrac{dy}{dx}-2xy=0$ $\quad y=3$ when $x=0$

m) $\dfrac{dy}{dx}=\dfrac{y+1}{x+4}$ $\quad y=3$ when $x=-2$

n) $\dfrac{dy}{dx}=y^2$ $\quad y=-1$ when $x=3$

o) $(x+1)\dfrac{dy}{dx}+2=y$ $\quad y=8$ when $x=1$

p) $\dfrac{dy}{dx}=\dfrac{y+2}{x+1}$ $\quad y=-2$ when $x=-1$

q) $x\dfrac{dy}{dx}+y^2=0\ (x>0)$ $\quad y=\dfrac{1}{2}$ when $x=1$

r) $(x+2)\dfrac{dy}{dx}+5=y$ $\quad y=1$ when $x=2$

2. a) Express $\dfrac{1}{y(y+1)}$ in partial fractions.

b) Hence solve the differential equation $\dfrac{dy}{dx}=\dfrac{y(y+1)}{x}$,

given that y = 4 when x = 2, expressing y explicitly in terms of x.

3. a) Express $\dfrac{2}{y^2-1}$ in partial fractions.

b) Hence solve the differential equation $2x\dfrac{dy}{dx}+1=y^2$,

given that y = -3 when x = 1, expressing y explicitly in terms of x.

4. a) Express $\dfrac{1}{(y+1)(y+2)}$ in partial fractions.

b) Hence show that the general solution of the differential equation $\dfrac{dy}{dx}=\dfrac{(y+1)(y+2)}{x}$

can be expressed in the form $y=\dfrac{1Ax-1}{1-Ax}$, where A is a constant.

Find the particular solution of this differential equation for which y = -3 when x = 1.

ANSWERS

1.

a) $y=\left(x^2+x+2\right)^{\frac{1}{3}}$

b) $y=\left(2x^3+3x-4\right)^{\frac{1}{3}}$

c) $y=\left(5x^3-6\right)^{\frac{1}{5}}$

d) $y=\ln(\cos x+1)$

e) $y=2e^x$

f) $y=4e^{2x^2}$

g) $y=3x-2$

h) $y=4(x+1)$

i) $y=x+1$

j) $y=4e^{3x^2}-1$

k) $y=2x+11$

l) $y=3e^{x^2}$

m) $y=2x+7$

n) $y=\dfrac{1}{2-x}$

o) $y=3x+5$

p) $y=5x+3$

q) $y=\dfrac{1}{\ln x+2}$

r) $y=3-x$

2. a) $\dfrac{1}{y(y+1)}=\dfrac{1}{y}-\dfrac{1}{y+1}$

b) $y=\dfrac{2x}{5-2x}$

3. a) $\dfrac{2}{y^2-1}=\dfrac{1}{y-1}-\dfrac{1}{y+1}$

b) $y=\dfrac{1+2x}{1-2x}$

4. a) $\dfrac{1}{(y+1)(y+2)}=\dfrac{1}{y+1}-\dfrac{1}{y+2}$

b) $y=\dfrac{4x-1}{1-2x}$

Advanced Higher Particular - Differential Equation 2 *Unit 1 Outcome 1.4*

1. a) Express $\dfrac{4}{y^2-4}$ in partial fractions.

   b) Hence show that the general solution of the differential equation $4\left(\dfrac{dy}{dx}+1\right)=y^2$

   can be expressed in the form $y=\dfrac{2\left(1+Ae^x\right)}{1-Ae^x}$, where A is a constant.

   Find the particular solution of this differential equation for which y = -4 when x = 0.

2. a) Express $\dfrac{2}{y(y+2)}$ in partial fractions.

   b) Hence solve the differential equation $2x\dfrac{dy}{dx}=y(y+2)$, given that y = 2 when x = 1, expressing y explicitly in terms of x.

3. At any point (x, y) on a curve with equation $y = f(x)$, the gradient of the tangent to the curve is given by $\dfrac{dy}{dx}=\dfrac{x^2+1}{y^2}$, $y\neq 0$.

   The point P(-1, 1) is on the curve.
   a) Find the equation of the tangent to the curve at P.
   b) Solve the differential equation to find the equation of the curve in the form $y = f(x)$.

4. a) Express $\dfrac{1}{x(x+1)}$ in partial fractions.

   b) At any point (x, y) on a curve with equation $y = f(x)$ where $x > 0$, the gradient of the tangent to the curve is given by $\dfrac{dy}{dx}=\dfrac{y}{x(x+1)}$,

   The point P(3, 6) is on the curve.

   i) Find the equation of the tangent to the curve at P.
   ii) Solve the differential equation to find the equation of the curve in the form $y = f(x)$.

5. a) Express $\dfrac{3}{x^2-x-2}$ in partial fractions.

   b) Hence find the general solution of the differential equation $\dfrac{dy}{dx}=\dfrac{3y}{x^2-x-2}$,

   in the region where $x > 2$, expressing y explicitly in terms of x.

   Find the particular solution of this differential equation for which y = 1 when x = 5.

ANSWERS

1. a) $\frac{4}{y^2-4}=\frac{1}{y-2}-\frac{1}{y+2}$ b) $y=\frac{2\left(1+3e^x\right)}{1-3e^x}$

2. a) $\frac{2}{y(y+2)}=\frac{1}{y}-\frac{1}{y+2}$ b) $y=\frac{2x}{2-x}$

3. a) $y=2x+3$ b) $y=\left(x^3+3x+5\right)^{\frac{1}{3}}$

4. a) $\frac{1}{x(x+1)}=\frac{1}{x}-\frac{1}{x+1}$ b) (i) $2y=x+9$ (ii) $y=\frac{8x}{x+1}$

5. a) $\frac{3}{x^2-x-2}=\frac{1}{x-2}-\frac{1}{x+1}$ b) $y=\frac{A(x-2)}{x+1}$; $y=\frac{2(x-2)}{x+1}$

MATHEMATICAL MODELS – GROWTH AND DECAY

1) The number of strands of Bacteria, x, present in a culture after t days of growth is assumed to be increasing at a rate proportional to the number of strands present.

a) Express this as a differential equation and find the general solution for x in terms of y.

b) If there are 326 strands initially present and 1833 strands are observed after 4 days, estimate the number of strands likely to be present after 1 week.

Read Scholar page 66 to 70
p70 Q57 - 60

2) Newton's Law of Cooling.
States that the rate at which an object cools is proportional to the difference in temperature between the object and the surrounding medium.

Surrounding Temp 75^{o}C $\quad \dfrac{dT}{dt} = -k(T - 75)$

T is the temp of object after t hours of cooling.
k = constant.

a) Find the general solution of this differential equation, expressing T as a function of t.

b) Given that a certain object cools from 125°C to 100°C in half an hour when surrounded by air at a temp of 75°C find

(i) the temperature of this object at the end of another half hour.

(ii) the time taken for the temperature of this object to fall to 80°C.

3) When a valve is opened the rate at which the water drains from a pool is proportional to the square root of the depth of the water. This can be represented by the d.e $\frac{dh}{dt} = -k\sqrt{h}$

where h metres is the depth of the water t minutes after the valve was opened and k is a positive constant.

a) Find the general solution of this differential equation. (You need not express h explicitly as a function of t).

b) Given that the pool was initially 9 metres deep and that the depth of water was 4 metres after 20 minutes of draining, how long did it take to drain the pool completely?

4) In a chemical reaction, two substances X and Y combine to form a third substance Z.
Let Q denote the number of grams of Z formed t minutes after the reaction begins.

The rate at which Q varies is represented by the d.e. $\frac{dQ}{dt} = \frac{(20-Q)(10-Q)}{400}$

a) Express $\frac{400}{(20-Q)(10-Q)}$ as a partial fraction.

b) Given that $Q = 0$ when $t = 0$, find the general solution of the differential equation, expressing Q explicitly as a function of t.

Worksheet – Mathematical Models

Scholar p75 Q61-65
p76 Q66 - 73

c) Find (i) the number of grams of Z formed one hour after the reaction begins,
and (ii) the time taken to form 5 grams of Z.

d) Find the limiting value of Q at $t \to \infty$

Scholar p75 Q61-65
p76 Q66 - 73

Worksheet – Mathematical Models

1. In a certain culture of bacteria, the rate of increase of bacteria at any given time is proportional to the number of bacteria present at that time. If x denotes the number of bacteria present after t hours of observation, the growth of bacteria in the culture is governed by a differential equation of the form

   $$\frac{dx}{dt} = kx,$$

   where k is a constant.

   a) Find the general solution of this differential equation, expressing x as a function of t.
   b) Given that the number of bacteria increased from 600 at time t = 0 to 1800 at time t = 2, find the number of bacteria present at the end of 4 hours (correct to the nearest whole number).

2. A scientist grows a culture of bacteria in his laboratory. The number of bacteria, N, after t hours is assumed to satisfy the differential equation $\frac{dN}{dt} = kN,$

   where k is a constant.

   There are initially 80 bacteria in the culture, and the scientist observes that there are 197 bacteria after 3 hour.

   How long will it take for the bacteria population to increase to 10 times the number initially present? (Give your answer correct to the nearest 0.1 hour).

3. A radioactive isotope of plutonium decays exponentially such that the rate of decrease of the mass of the isotope at any given time is proportional to the mass remaining at that time. If m denotes the mass of the isotope (in grams) remaining t years after decay began, the decay is governed by a differential equation of the form $\frac{dm}{dt} = -km,$

   where k is a positive constant.

   a) Find the general solution of this differential equation, expressing m as a function of t.
   b) Given that this isotope has a half-life of approximately 139 years (that is, given any initial mass of the isotope, one half will disintegrate in 139 years), show that starting with 120 grams of this isotope, the mass remaining after t years is given approximately by $m = 120e^{-0.005t}$

      Hence find the mass remaining after 50 years.

4. A rumour is spread among a large population of students. It is assumed that the rate at which the rumour is spreading is proportional to the number of students who have heard the rumour. Hence, if N is the number of students who have heard the rumour after t days, N satisfies a differential equation of the form $\frac{dN}{dt} = kN,$

   where k is a constant.

   The rumour is spread initially by 2 students and, after five days 985 students had heard the rumour.

   a) Solve this differential equation and show that, approximately, $N = 2e^{1.24t}$.

   b) After how many days had 250 students heard that rumour?
      (Give your answer to the nearest 0.1day)

5. A television producer predicts that viewing figures for her programme will increase at a rate proportional to the number of viewers. Hence, if n is the number of viewers after t weeks, it can be assumed that n satisfies a differential equation of the form $\dfrac{dn}{dt} = kn$,

   where k is a constant.

   a) Find the general solution of this differential equation, expressing n as a function of t.

   The number of viewers at the moment (time $t = 0$) is 100 000 and the producer predicts that there will be 120 925 viewers in two weeks' time.

   b) (i) Find the value of the constant k correct to three decimal places.
   (ii) The producer is told that the target audience for her programme is 500 000 viewers. How long should it take to reach this target, to the nearest 0.1 week?

6. A metal plate that has been heated cools from 180°C to 150°C in 20 minutes when surrounded by air at a temperature of 60°C. Newton's Law of Cooling states that if T is the temperature of the plate (in °C) after t minutes of cooling, then T satisfies a differential equation of the form $\dfrac{dT}{dt} = k(T - 60)$,

   where k is a constant.

   a) Find the temperature of the plate at the end of one hour of cooling, correct to the nearest 0.1°C.
   b) When will the temperature of the plate reach 100°C, correct to the nearest minute?

7. If the temperature is constant, then the rate of change of barometric pressure p with respect to altitude h is proportional to p, where p and h are measured in appropriate units. The barometric pressure is thus governed by a differential equation of the form $\dfrac{dp}{dh} = kp$,

   where k is a constant.

   If $p = 30$ at sea level ($h = 0$) and $p = 29$ when $h = 1$, find p when $h = 5$, correct to the nearest 0.1 unit.

8. In a certain culture of bacteria, the rate of increase of bacteria at any given time is proportional to the number of bacteria present at that time. Thus if x denotes the number of bacteria present after t hours of observation, then x satisfies a differential equation of the form $\dfrac{dx}{dt} = kx$,

   where k is a constant.

   Given that the number of bacteria increased from 5000 (at time $t = 0$) to 15 000 in 10 hours, estimate the number of bacteria at the end of 20 hours, giving your answer correct to the nearest thousand.

   How many hours will it take for the number of bacteria to grow to 50 000?
   (Give your answer correct to the nearest 0.1 hour).

9. The polonium isotope ^{210}Po decays such that the rate of decrease of the mass of this isotope at any given time is proportional to the mass remaining. If *m* denotes the mass of this isotope (in mg) remaining after *t* days of decay, the decay is modelled by a differential equation of the form

$\frac{dm}{dt} = -km,$ where *k* is a positive constant.

Given that the isotope has a half-life (see question 3) of approximately 140 days, estimate the mass which will remain from a 20mg sample after 2 weeks, giving your answer correct to the nearest 0.1mg.

10. The rate at which a body loses speed at any given time *t* seconds as it travels through a resistive medium is given by *kv* metres per second per second, where *v* metres per second is the speed of the body at that instant, and *k* is a positive constant. The speed of this body is governed by the differential equation $\frac{dv}{dt} = -kv,$

a) Find the general solution of this differential equation, expressing *v* as a function of *t*.
b) Show that if the initial speed of the body is *u* metres per second, then the time taken for the body to decrease its speed to $\frac{1}{2}u$ metres per second is $\frac{1}{k}\ln 2$ seconds.

11. The rate at which a population of mice is increasing at any given time is assumed to be proportional to the number of mice.

a) If *m* denotes the number of mice after t months, write down a differential equation satisfied by m and find the general solution, expressing m as a function of *t*.
b) Given that at time $t = 0$ there are 6 mice and it takes 6 months for the number of mice to double, after how many months will there be 90 mice (correct to the nearest 0.1 month)?

12. When taken from the oven, the temperature, T°C, of a bowl of pasta after *t* minutes of cooling is such that $\frac{dT}{dt} = -kT,$
where *k* is a positive constant.

The bowl of pasta was at a temperature of 70°C after one minute of cooling, and was at a temperature 49°C after a further minute of cooling.

a) Find the temperature of the pasta on leaving the oven, correct to the nearest °C.
b) How long, to the nearest 0.1minute, will the pasta take to cool down to the room temp of 20°C?

13. When a valve is opened, the rate at which the water drains from a pool is proportional to the square root of the depth of the water. This can be represented by the differential equation $\frac{dh}{dt} = -k\sqrt{h},$

where *h* is the depth (in metres) of the water and *t* is the time (in minutes) elapsed since the valve was opened, and *k* is a positive constant.

a) Find the general solution of the differential equation. (You need not express *h* explicitly as a function of *t*).
b) Given that the pool was initially 4 metres deep, and that the depth of water was 1 metre after 20 minutes of draining, how long did it take for the pool to drain completely?

14. In a town with a large population, a flu virus is spreading rapidly. The percentage P infected t days after the initial outbreak satisfies the differential equation
$$\frac{dP}{dt} = kP,$$
where k is a constant.

0.5% of the population are infected initially, and 2.5% are infected after 7 days.

Find the value of k correct to 4 decimal places and verify that about 12.5% of the population will be infected after 14 days.

15. a) Express $\dfrac{1}{x(1-x)}$ in partial fractions.

b) The spread of a disease in a large population can be modelled by means of a differential equation. The proportion x of the population infected with the disease after t days satisfies a differential equation of the form
$$\frac{dx}{dt} = kx(1-x),$$
where k is a constant.

Find the general solution of the differential equation, expressing x explicitly as a function of t.

Given that $x = \dfrac{1}{1000}$ when $t = 0$ and $x = \dfrac{1}{100}$ when $t = 5$, verify that about 9% of the population was infected after ten days, and find after how many days 25% of the population will become infected.

16. In a chemical reaction, two substances X and Y combine to form a third substance Z. Let Q denote the number of grams of Z formed t minutes after the reaction begins. The rate at which Q varies is represented by the differential equation

a) Express $\dfrac{900}{(30-Q)(15-Q)}$ in partial fractions.

b) Use your answer to (a) to find the general solution of the differential equation, expressing Q explicitly as a function of t.

Given that $Q = 0$ when $t = 0$, find correct to two decimal places:
(i) the number of grams of Z formed 45 minutes after the reaction begins
(ii) the time taken to form 5 grams of Z.

17. a) Express $\dfrac{1}{p(1-p)}$ in partial fractions.

b) The spread of a disease in a large population can be modelled by means of a differential equation. The proportion p of the population infected with the disease after t days satisfies a differential equation of the form

$p = \dfrac{1}{500}$ when $t = 0$, and $p = \dfrac{1}{100}$ when $t = 2$. $$\frac{dp}{dt} = kp\left(1-p\right)$$

(i) Express p explicitly as a function of t.
(ii) Verify that about 10% of the population was infected after five days.
(iii) How long will it take for 50% of the population to become infected?

ANSWERS

1. a) $x = Ae^{kt}$ b) 5400

2. 7.7 hours

3. a) $m = Ae^{-kt}$ b) 93.5 grams

4. b) 3.9 days

5. a) $n = Ae^{kt}$ b) (i) $k = 0.095$ (ii) 16.9 weeks

6. a) 110.6°C b) 76 mins (or 1 hour 16 mins)

7. $p = 25.3$

8. 45,000; 21.0 hours

9. 18.6 mg

10 a) $v = Ae^{-kt}$

11 a) $\frac{dm}{dt} = km$ where k is a constant b) 23.4 months

12 a) 100°C b) 4.5 mins

13 a) $2\sqrt{h} = C - kt$ b) 40 mins

14 $k = 0.2299$

15 a) $\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$ b) $x = \frac{Ae^{kt}}{1+Ae^{kt}}$;

16 a) $\frac{900}{(30-Q)(15-Q)} = \frac{60}{30-Q} + \frac{60}{15-Q}$

b) $Q = \frac{15\left(Ae^{\frac{t}{60}} - 2\right)}{Ae^{\frac{t}{60}} - 1}$; (i) 10.36 grams (ii) 13.39 mins

17 a) $\frac{1}{p(1-p)} = \frac{1}{p} + \frac{1}{1-p}$

b) (i) $p = \frac{e^{0.8087t}}{499+e^{0.8087t}}$; (ii) 7.7 days

Reminder: A differential equation is an equation involving one or more derivatives.

Example

First Derivative

$$\left(x^2-1\right)\frac{dy}{dx}+2xy=x$$

First Order Differential Equation

Second Derivative

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+4y=0$$

Second Order Differential Equation

A Linear first order differential equation has only $\frac{dy}{dx}$ and y (not y^2, $y\frac{dy}{dx}$, y^3 or $\cos y$ etc.)

A first order differential equation can be written in the form

$$\boxed{\frac{dy}{dx}+P(x)y=f(x)}$$

$P(x)$ and $f(x)$ are functions of x only.

(does not include y^2, e^y, $\sin y$ etc.)

To find the general solution, first find the **INTEGRATING FACTOR** $I(x)$.

INTEGRATING FACTOR $I(x)=e^{\int P(x)dx.}$

Once the integrating factor has been found it is then used to calculate the general solution

$$\boxed{I(x)y=\int I(x).f(x).dx}$$

Steps to find the General Solution:

1. Write in the form $\frac{dy}{dx} + P(x)y = f(x)$ and identify $P(x)$ = $f(x)$ =

2. Find $I(x) = e^{\int P(x)dx}$.

3. Substitute $I(x)$ into $I(x)y = \int I(x).f(x).dx$

4. Integrate

5. Rearrange to solve for y.

NOTE: the form must be $\frac{dy}{dx}$ singular and not $3\frac{dy}{dx}$.

Example 1.

Find the general solution of $\frac{dy}{dx} + y = 6e^{2x}$.

Solution

Example 2.

Find the general solution of $\frac{dy}{dx}+\frac{y}{x}=x.$

Solution

Example 3.

Find the general solution of $x\frac{dy}{dx}-2y=x^3\sin x.$

Solution

Worksheet 1st ODE - 1

Scholar page 131 Ex1 Q7 - 12

1. Find the general solution of each differential equation, in each case expressing y **explicitly** in terms of x.

a) $\dfrac{dy}{dx}+2y=1$ b) $\dfrac{dy}{dx}+y=2e^{x}$ c) $\dfrac{dy}{dx}+2y=e^{2x}$

d) $\dfrac{dy}{dx}-y=-2e^{-x}$ e) $\dfrac{dy}{dx}+2y=e^{-x}$ f) $\dfrac{dy}{dx}-2y=6e^{-x}$

g) $\dfrac{dy}{dx}+\dfrac{y}{x}=4x^2$ h) $\dfrac{dy}{dx}+\dfrac{2y}{x}=x$ i) $\dfrac{dy}{dx}-\dfrac{2y}{x}=2$

j) $\dfrac{dy}{dx}-2y=e^{3x}$ k) $\dfrac{dy}{dx}+2xy=e^{-x^2}\cos x$ l) $\dfrac{dy}{dx}+\dfrac{y}{x}=x$

2. Rearrange each differential equation into the standard form $\dfrac{dy}{dx}+P(x).y=Q(x)$ and hence find the general solution of each equation., expressing y **explicitly** in terms of x.

a) $e^x\dfrac{dy}{dx}+e^x y=\cos 2x$ b) $x^2\dfrac{dy}{dx}-xy=4$ c) $x\dfrac{dy}{dx}+y=x^2$

d) $x\dfrac{dy}{dx}+y=\sin x$ e) $x\dfrac{dy}{dx}+2y=\dfrac{4}{x}$ f) $x\dfrac{dy}{dx}-2y=6x^4$

g) $2\dfrac{dy}{dx}+y=e^x$ h) $3x\dfrac{dy}{dx}+3y=2x$ i) $x\dfrac{dy}{dx}-2y=x$

j) $x^2\dfrac{dy}{dx}+2xy=\cos x$ k) $x\dfrac{dy}{dx}-2y=-3x$

3. a) Use the method of integration by parts to find $\int x\sin x\,dx$.

b) Find the general solution of the differential equation $x\dfrac{dy}{dx}+y=x\sin x$, expressing y explicitly in terms of x.

4. a) By writing tan x as $\dfrac{\sin x}{\cos x}$ and using a suitable substitution, show that $\int \tan x\,dx=\ln(\sec x)+C$.

b) Find the general solution of the differential equation $\cos x\dfrac{dy}{dx}+y\sin x=\cos^2 x$, expressing y explicitly in terms of x.

5. a) By writing cot x as $\dfrac{\cos x}{\sin x}$ and using a suitable substitution, show that $\int \cot x\,dx=\ln(\sin x)+C$.

b) Find the general solution of the differential equation $\dfrac{dy}{dx}+y\cot x=\cot x$, expressing y explicitly in terms of y.

ANSWERS

1.

a) $y=\frac{1}{2}+\frac{C}{e^{2x}}$ b) $y=e^{x}+\frac{C}{e^{x}}$ c) $y=\frac{1}{4}e^{2x}+\frac{C}{e^{2x}}$

d) $y=e^{-x}+Ce^{x}$ e) $y=\frac{1}{e^{x}}+\frac{C}{e^{2x}}$ f) $y=-2e^{-x}+Ce^{2x}$

g) $y=x^{3}+\frac{C}{x}$ h) $y=\frac{1}{4}x^{2}+\frac{C}{x^{2}}$ i) $y=-2x+Cx^{2}$

j) $y=e^{3x}+Ce^{2x}$ k) $y=\frac{\sin x+C}{e^{x^{2}}}$ l) $y=\frac{1}{3}x^{2}+\frac{C}{x}$

2.

a) $y=\frac{\sin 2x}{2e^{x}}+\frac{C}{e^{x}}$ b) $y=-\frac{2}{x}+Cx$ c) $y=\frac{1}{3}x^{2}+\frac{C}{x}$

d) $y=\frac{C-\cos x}{x}$ e) $y=\frac{4}{x}+\frac{C}{x^{2}}$ f) $y=3x^{4}+Cx^{2}$

g) $y=\frac{1}{3}e^{x}+\frac{C}{e^{\frac{1}{2}x}}$ h) $y=\frac{1}{3}x+\frac{C}{x}$ i) $y=-x+Cx^{2}$

j) $y=\frac{\sin x+C}{x^{2}}$ k) $y=3x+Cx^{2}$

3. a) $\int x\sin x\,dx=\sin x-x\cos x+C$

b) $y=\frac{\sin x}{x}-\cos x+\frac{C}{x}$

4. b) $y=(x+C)\cos x$

5. b) $y=1+\frac{C}{\sin x}$

PARTICULAR SOLUTION OF 1ST ORDER D.E

The examples so far have solved 1st Order differential equations finding the General Solution.

If an initial condition is known it is possible to find the value of constant C and therefore the PARTICULAR SOLUTION.

Example 1

Find the particular solution of the differential equation $x\frac{dy}{dx}+3y=5x^2$ given that $y = 3$ when $x = 1$.

Solution

Worksheet 1st ODE - 2

Scholar page 131 Ex1 Q13 - 17

1. Find the particular solution of each differential equation subject to the given condition in brackets.

a) $\frac{dy}{dx}+y=1$ $\quad (y=3 \text{ when } x=0)$

b) $x\frac{dy}{dx}+3y=4x$ $\quad (y=4 \text{ when } x=1)$

c) $x\frac{dy}{dx}+3y=5x^2$ $\quad (y=0 \text{ when } x=1)$

d) $x^2\frac{dy}{dx}+2xy=1$ $\quad (y=2 \text{ when } x=1)$

e) $x\frac{dy}{dx}-y=x^2$ $\quad (y=0 \text{ when } x=1)$

f) $x\frac{dy}{dx}-y=x^2\cos x$ $\quad \left(y=0 \text{ when } x=\frac{\pi}{2}\right)$

g) $\frac{dy}{dx}-y\cos x=2xe^{\sin x}$ $\quad (y=1 \text{ when } x=0)$

h) $x\frac{dy}{dx}+2y=6x$ $\quad (y=1 \text{ when } x=1)$

i) $\frac{dy}{dx}+3x^2y=6x^2$ $\quad (y=1 \text{ when } x=0)$

j) $x\frac{dy}{dx}+2y=6x^4$ $\quad (y=5 \text{ when } x=1)$

2. a) Use the method of integration by parts to find $\int xe^x dx$.

b) Find the particular solution of the differential equation $\frac{dy}{dx}+y=x$, given that $y = 1$ when $x = 0$.

3. a) By writing $\tan x$ as $\frac{\sin x}{\cos x}$ and using a suitable substitution, show that $\int \tan x dx = \ln(\sec x)+C$

b)(i) Show that the general solution of the differential equation $\frac{dy}{dx}+y\tan x=\sec x$,

can be written in the form $y=\sin x+C\cos x$, where C is a constant.

(ii) Find the particular solution of this differential equation given that $y = 1$ when $x = 0$.

4. a) Use the method of integration by parts to find $\int \ln x dx$.

b) Find the particular solution of the differential equation $x\frac{dy}{dx}+y=\ln x$, given that $y = 1$ when $x = 1$.

5. a) By writing $\tan x$ as $\frac{\sin x}{\cos x}$ and using a suitable substitution, show that $\int \tan x dx = \ln(\sec x)+C$

b)(i) Show that the general solution of the differential equation $\frac{dy}{dx}+y\tan x=\cos^2 x$,

can be written in the form $y=(\sin x+C)\cos x$, where C is a constant.

(ii) Find the particular solution of this differential equation given that $y=\frac{3}{2}$ when $x=\frac{\pi}{4}$.

ANSWERS

1.

a) $y = 1 + \frac{2}{e^x}$

b) $y = x + \frac{3}{x^3}$

c) $y = x^2 - \frac{1}{x^3}$

d) $y = \frac{1}{x} + \frac{1}{x^2}$

e) $y = x^2 - x$

f) $y = x\sin x - x$

g) $y = \left(x^2 + 1\right)e^{\sin x}$

h) $y = 2x - \frac{1}{x^2}$

i) $y = 2 - \frac{1}{e^{x^3}}$

j) $y = x^4 + \frac{4}{x^2}$

2.a) $\int xe^x dx = xe^x - e^x + C$

b) $y = x - 1 + \frac{2}{e^x}$

3. b)(ii) $y = \sin x + \cos x$

4. a) $\int \ln x dx = x\ln x - x + C$

b) $y = \ln x - 1 + \frac{2}{x}$

5. b)(ii) $y = \left(\sin x + \sqrt{2}\right)\cos x$

Applications of 1st Order Differential Equations

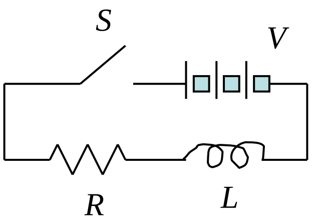
Example 1

Bacteria grows at a rate proportional to the amount of bacteria present.
If there are 100 bacteria present initially:

- Find an equation for the bacterial growth
- Find the number of bacteria present at any given time by solving the differential equation.

Solution

Example 2



An electical circuit show has a constant DC voltage of 12V
The circuit has a switch, S, Inductor L and a resistor R of value 6 Ohms.

Current = $i(t)$ where the time is in seconds after the switch is closed.

$$L\frac{di}{dt} + Ri = V \qquad i(0) = 0$$

Find a formula for $i(t)$ in terms of L R and V.

Solution

Example 3

A tank contains 50kg of salt dissolved in 200m^3 of water.
Starting at time $t = 0$, water which contains 2 kg of salt per m^3 enters the tank at a rate of 4m^3/min and leaves at the same rate.

Calculate how much salt there is after 30mins.

Solution

Example 3

A tank contains 50kg of salt dissolved in 200m^3 of water.
Starting at time $t = 0$, water which contains 2 kg of salt per m^3 enters the tank at a rate of 4m^3/min and leaves at the same rate.

Calculate how much salt there is after 30mins.

Solution continued

Second Order Differential Equations

A second order D.E. is expressed

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

a, b and c are constants where a ≠ 0.

$f(x)$ is a function of only x.

IF $f(x) = 0$	then it is **HOMOGENEOUS**
IF $f(x) \neq 0$	then it is **NON-HOMOGENEOUS**.

Homogeneous 2nd Order Differential Equations

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

To find the General Solution:

1. Form a quadratic "**auxiliary equation**". $ak^2+bk+c=0$
2. Find the roots of the auxiliary equation.
3. The nature of the general solution depends on the roots:

A) If the roots are real and distinct ($b^2 - 4ac > 0$) where $k = p$ and $k = q$ and $p \neq q$ then

$$\boxed{y = Ae^{px} + Be^{qx}}$$

A and B are constants.

B) If the roots are real and **equal** ($b^2 - 4ac = 0$) where $k = p$ and $p = q$ then

$$\boxed{y=(Ax+B)e^{px}}$$

A and B are constants.

C) If the roots are complex conjugates ($b^2 - 4ac < 0$) where $k = p \pm qi$ then

$$\boxed{y=e^{px}(A\sin qx+B\cos qx)}$$

A and B are constants.

Example 1

Find the general solution of the 2nd order d.e. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 0$

Example 2

Find the general solution of the 2nd order d.e. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

Example 3

Find the general solution of the 2nd order d.e. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

Worksheet – 2nd Order Homogeneous D.E. 1

Scholar Page 142 Ex3 Q24 - 35

1. Find the general solution of each differential equation.

a) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+3y=0$

b) $\frac{d^2y}{dx^2}-\frac{dy}{dx}-6y=0$

c) $\frac{d^2y}{dx^2}-\frac{dy}{dx}-2y=0$

d) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$

e) $\frac{d^2y}{dx^2}-10\frac{dy}{dx}+25y=0$

f) $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=0$

g) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+5y=0$

h) $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+13y=0$

i) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0$

j) $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$

k) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}=0$

l) $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+16y=0$

m) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+13y=0$

n) $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=0$

o) $\frac{d^2y}{dx^2}-16y=0$

p) $2\frac{d^2y}{dx^2}+5\frac{dy}{dx}-3y=0$

q) $9\frac{d^2y}{dx^2}-30\frac{dy}{dx}+25y=0$

r) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+8y=0$

s) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=0$

t) $\frac{d^2y}{dx^2}-7\frac{dy}{dx}+12y=0$

u) $2\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=0$

v) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+8y=0$

w) $4\frac{d^2y}{dx^2}-9y=0$

x) $9\frac{d^2y}{dx^2}+12\frac{dy}{dx}+4y=0$

y) $2\frac{d^2y}{dx^2}-2\frac{dy}{dx}+5y=0$

z) $6\frac{d^2y}{dx^2}+17\frac{dy}{dx}+5y=0$

ANSWERS

1.

a) $y = Ae^{x} + Be^{3x}$

b) $y = Ae^{3x} + Be^{-2x}$

c) $y = Ae^{2x} + Be^{-x}$

d) $y = (Ax + B)e^{2x}$

e) $y = (Ax + B)e^{5x}$

f) $y = (Ax + B)e^{-3x}$

g) $y = e^{x}(A\sin 2x + B\cos 2x)$

h) $y = e^{3x}(A\sin 2x + B\cos 2x)$

i) $y = e^{-2x}(A\sin x + B\cos x)$

j) $y = Ae^{2x} + Be^{3x}$

k) $y = A + Be^{2x}$

l) $y = (Ax + B)e^{4x}$

m) $y = e^{-2x}(A\sin 3x + B\cos 3x)$

n) $y = Ae^{-2x} + Be^{-3x}$

o) $y = Ae^{4x} + Be^{-4x}$

p) $y = Ae^{\frac{1}{2}x} + Be^{-3x}$

q) $y = (Ax + B)e^{\frac{5}{3}x}$

r) $y = e^{2x}(A\sin 2x + B\cos 2x)$

s) $y = e^{-x}(A\sin x + B\cos x)$

t) $y = Ae^{3x} + Be^{4x}$

u) $y = Ae^{\frac{3}{2}x} + Be^{-2x}$

v) $y = e^{-2x}(A\sin 2x + B\cos 2x)$

w) $y = Ae^{\frac{3}{2}x} + Be^{-\frac{3}{2}x}$

x) $y = (Ax + B)e^{-\frac{2}{3}x}$

y) $y = e^{\frac{1}{2}x}\left(A\sin\frac{3}{2}x + B\cos\frac{3}{2}x\right)$

z) $y = Ae^{-\frac{1}{3}x} + Be^{-\frac{5}{2}x}$

PARTICULAR HOMOGENEOUS SOLUTIONS

By using initial conditions the constants A and B can be found therefore the particular solution.

Example 1

Find the particular solution of the 2nd order d.e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$ given that when $x = 0$, $y = 1$ and $\frac{dy}{dx} = 5$.

PARTICULAR HOMOGENEOUS SOLUTIONS

Example 2

Find the particular solution of the 2nd order d.e. $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$ given that when x = 0, y = 1 and $\frac{dy}{dx}=2$.

Example 3

Find the particular solution of the 2nd order d.e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$ given that when $x = 0$, $y = 2$ and $\frac{dy}{dx} = 1$.

Worksheet – 2nd Order Homogeneous D.E. 2

Scholar Page 145 Ex4 Q36 - 44

1. Find the particular solution of each differential equation.

a) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ when $x = 0,\ y = 0,\ \frac{dy}{dx} = 2.$

b) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$ when $x = 0,\ y = 0,\ \frac{dy}{dx} = 10.$

c) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$ when $x = 0,\ y = 3,\ \frac{dy}{dx} = 5.$

d) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$ when $x = 0,\ y = 6,\ \frac{dy}{dx} = 0.$

e) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$ when $x = 0,\ y = 2,\ \frac{dy}{dx} = 9.$

f) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ when $x = 0,\ y = 1,\ \frac{dy}{dx} = 2.$

g) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ when $x = 0,\ y = 1,\ \frac{dy}{dx} = 3.$

h) $\frac{d^2y}{dx^2} - y = 0$ when $x = 0,\ y = 1,\ \frac{dy}{dx} = 2.$

i) $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0$ when $x = 0,\ y = 3,\ \frac{dy}{dx} = 20.$

j) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 10y = 0$ when $x = 0,\ y = 2,\ \frac{dy}{dx} = 1.$

k) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 0$ when $x = 0,\ y = 5,\ \frac{dy}{dx} = 2.$

l) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ when $x = 0,\ y = 0,\ \frac{dy}{dx} = 3.$

m) $\frac{d^2y}{dx^2} + 9y = 0$ when $x = 0,\ y = 10,\ \frac{dy}{dx} = 0.$

n) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$ when $x = 0,\ y = 2,\ \frac{dy}{dx} = 3.$

o) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ when $x = 0,\ y = \frac{1}{4},\ \frac{dy}{dx} = -1.$

p) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$ when $x = 0,\ y = 2,\ \frac{dy}{dx} = 0.$

ANSWERS

1.

a) $y = 2e^{2x} - 2e^{x}$

b) $y = 2e^{2x} - 2e^{-3x}$

c) $y = 2e^{x} + e^{3x}$

d) $y = 4e^{x} + 2e^{-2x}$

e) $y = (3x+2)e^{3x}$

f) $y = (4x+1)e^{-2x}$

g) $y = e^{x}(2\sin x + \cos x)$

h) $y = 2e^{x} - 1$

i) $y = (5x+3)e^{5x}$

j) $y = e^{3x}(-5\sin x + 2\cos x)$

k) $y = 2e^{4x} + 3e^{-2x}$

l) $y = 3xe^{2x}$

m) $y = 10\cos 3x$

n) $y = e^{-x}(5\sin x + 2\cos x)$

o) $y = \left(-\frac{3}{4}x + \frac{1}{4}\right)e^{-x}$

p) $y = e^{-2x}\left(\frac{4}{3}\sin 3x + 2\cos 3x\right)$

Non-Homogeneous 2nd Order Differential Equations

$$\boxed{a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)}$$

where $f(x)$ is a simple function.

Steps to solve Non Homogeneous DE

1. Find the general solution of the homogeneous d.e. $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$

   This is called the **COMPLEMENTARY FUNCTION.**

2. Find a particular solution of the non-homogeneous. This is called the **PARTICULAR INTEGRAL**.

3. **GENERAL SOLUTION OF NON-HOMOGENEOUS = COMPLEMENTARY FUNCTION + PARTICULAR INTEGRAL**

Finding the Particular Integral

1) If $f(x)=2x+1$ $\qquad p.i. \rightarrow y=px+q$

2) If $f(x)=x^2-1$ $\qquad p.i. \rightarrow y=px^2+qx+r$

3) If $f(x)=4e^{2x}$ $\qquad p.i. \rightarrow y=pe^{2x}$

4) If $f(x)=2\sin x+\cos x$ $\qquad p.i. \rightarrow y=p\sin x+q\cos x$

5) If $f(x)=3\sin 2x$ $\qquad p.i. \rightarrow y=p\sin 2x+q\cos 2x$

Example 1

Find the general solution of the 2nd order d.e. $\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}-2y=4x+1$

Example 2

Find the general solution of the 2nd order d.e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2x^2 + 1$

Worksheet – 2nd Order Non-Homogeneous D.E. 1

1. Find the general solution of each differential equation (the particular integral will be a polynomial function in each case).

a) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 6x + 1$

b) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 12x$

c) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 8x^2 + 3$

d) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = 20x + 4$

e) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 18$

f) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 12x + 3$

g) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 8 - 10x$

h) $\frac{d^2y}{dx^2} + 4y = 3x$

i) $\frac{d^2y}{dx^2} + 16y = 4x^2$

j) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = x$

k) $4\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 2y = 1 - x^2$

ANSWERS

1.

a) $y = Ae^{2x} + Be^{3x} + x + 1$

b) $y = Ae^{2x} + Be^{-x} - 6x + 3$

c) $y = Ae^{x} + Be^{4x} + 2x^2 + 5x + 6$

d) $y = Ae^{2x} + Be^{-5x} - 2x - 1$

e) $y = (Ax + B)e^{3x} + 2$

f) $y = Ae^{4x} + Be^{-x} - 3x + \frac{3}{2}$

g) $y = e^{-x}(A\sin 3x + B\cos 3x) + 1 - x$

h) $y = A\sin 2x + B\cos 2x + \frac{3}{4}x$

i) $y = A\sin 4x + B\cos 4x + \frac{1}{4}x^2 - \frac{1}{32}$

j) $y = Ae^{3x} + Be^{-4x} - \frac{1}{12}x - \frac{1}{144}$

k) $y = Ae^{-\frac{1}{2}x} + Be^{-x} - \frac{1}{2}x^2 + 3x - \frac{13}{2}$

Example 3

Find the general solution of the 2nd order d.e. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 10e^{2x}$.

Example 4

Find the general solution of the 2nd order d.e. $\frac{d^2y}{dx^2}+16y=e^{-2x}$.

Worksheet – 2nd Order Non-Homogeneous D.E. 2

1. Find the general solution of each differential equation (the particular integral will be a polynomial function in each case).

a) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 4e^{2x}$

b) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 8e^{3x}$

c) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 10e^{2x}$

d) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 4e^{-x}$

e) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{4x}$

f) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10e^{3x}$

g) $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = e^{-2x}$

h) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = e^{-x}$

i) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = e^{2x}$

j) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 17y = 10e^{x}$

k) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-3x}$

ANSWERS

1.

a) $y = Ae^{x} + Be^{-2x} + e^{2x}$

b) $y = Ae^{2x} + Be^{-x} + 2e^{3x}$

c) $y = Ae^{x} + Be^{-3x} + 2e^{2x}$

d) $y = Ae^{x} + Be^{-2x} - 2e^{-x}$

e) $y = (Ax + B)e^{3x} + e^{4x}$

f) $y = e^{-x}(A\sin 2x + B\cos 2x) + \frac{1}{2}e^{3x}$

g) $y = Ae^{-3x} + Be^{-4x} + \frac{1}{2}e^{-2x}$

h) $y = Ae^{3x} + Be^{-2x} - \frac{1}{4}e^{-x}$

i) $y = Ae^{x} + Be^{-3x} + \frac{1}{5}e^{2x}$

j) $y = e^{-x}(A\sin 4x + B\cos 4x) + \frac{1}{2}e^{x}$

k) $y = e^{-x}(A\sin 2x + B\cos 2x) + \frac{1}{8}e^{-3x}$

Example 5

Find the general solution of the 2nd order d.e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 10\sin x.$

Example 6 **(Exam Question)**

Find the particular solution of the 2nd order d.e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x$, given that when $x = 0$, $y = 2$ and $\frac{dy}{dx} = 1$.

Occasionally the form of the p.i must be amended.

The Particular Integral (right hand side of equation) cannot be in the same form as the complementary function.

When working out the Particular Integral the left hand side will cancel out to give zero.

$y = Ae^{3x} + Be^{2x}$ $PI = 2x$ PI is not in the same form therefore $y = px + q$

$y = Ae^{3x} + Be^{2x}$ $PI = 2e^{4x}$ PI is not in the same form therefore $y = pe^{4x}$

$y = Ae^{3x} + Be^{2x}$ $PI = 2e^{3x}$ PI is in the same form (e^{3x}) therefore $y = pxe^{3x}$

$y = A + Be^{2x}$ $PI = 2x + 3$ PI is in the same form (A and 3) therefore $y = x(px + q)$

$y = A\sin 3x + B\cos 3x$ $PI = 4\sin 3x$

PI is in the same form therefore $y = px\sin 3x + qx\cos 3x$

$y = Ae^{2x} + Bxe^{2x}$ $PI = 5e^{2x}$ PI is in the same form (e^{2x} *and* xe^{2x}) therefore $y = px^2e^{2x}$

Example 7

Find the general solution of the 2nd order d.e. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2e^{3x}$.

Worksheet – 2nd Order Non-Homogeneous D.E. 3

Scholar Page 148 Ex4 Q45-65

1. Find the general solution of each differential equation (the particular integral will be a trigonometric function in each case).

a) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 40\sin x$

b) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 3\sin x + 19\cos x$

c) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 25\sin x$

d) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = -\sin x$

e) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 5\sin x$

f) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = \sin 2x$

g) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = \sin x$

h) $2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2\cos x$

ANSWERS

1.

a) $y = Ae^{x} + Be^{3x} + 4\sin x + 8\cos x$

b) $y = Ae^{x} + Be - 2^{3x} + \sin x - 6\cos x$

c) $y = (Ax + B)e^{-2x} + 2\sin x - 4\cos x$

d) $y = e^{x}(A\sin x + B\cos x) + \frac{1}{5}\sin x + \frac{2}{5}\cos x$

e) $y = e^{-x}(A\sin x + B\cos x) + \sin x - 2\cos x$

f) $y = e^{x}(A\sin x + B\cos x) - \frac{1}{10}\sin 2x + \frac{1}{5}\cos 2x$

g) $y = Ae^{2x} + Be^{-3x} - \frac{7}{50}\sin x - \frac{1}{50}\cos x$

h) $y = e^{\frac{1}{2}x}\left(A\sin\frac{1}{2}x + B\cos\frac{1}{2}x\right) - \frac{4}{5}\sin x - \frac{2}{5}\cos x$

1. Find the general solution of each differential equation (the particular solution will be an amended exponential function in each case).

a) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$

b) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 3e^{2x}$

c) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 4e^{3x}$

d) $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = e^{-3x}$

e) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 6e^{4x}$

ANSWERS

1.

a) $y = Ae^{x} + Be^{2x} - xe^{x}$

b) $y = Ae^{2x} + Be^{3x} - 3xe^{2x}$

c) $y = Ae^{2x} + Be^{3x} - \frac{4}{9}xe^{3x}$

d) $y = Ae^{-3x} + Be^{-4x} + xe^{-3x}$

e) $y = (Ax + B)e^{4x} + 3x^{2}e^{4x}$