Calculus skills (1.3)

- integrating expressions using standard results
- integrating by substitution
- integrating by parts

STANDARD INTEGRALS

Recap Integration from Higher.

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C, \qquad n \neq -1$$

$$\int (ax+b)^n dx = \left(\frac{1}{a}\right)\left(\frac{1}{n+1}\right)(ax+b)^{n+1} + C, \quad n \neq -1$$

Cannot integrate if power is -1.

Remember to add constant of integration, C.

Examples

1) $\int (2x+3)^5 dx$

$$= \frac{1}{6}(2x+3)^6\left(\frac{1}{2}\right) + C$$

$$= \frac{1}{12}(2x+3)^6 + C$$

2) $\int \frac{3x^2 - 1}{\sqrt{x}} dx$

$$= \int \frac{3x^2}{x^{1/2}} - \frac{1}{x^{1/2}} dx$$

$$= \int 3x^{3/2} - x^{-1/2} dx$$

$$= 3\left(\frac{2}{5}\right)x^{5/2} - 2x^{1/2} + C$$

$$= \frac{6}{5}x^{5/2} - 2\sqrt{x} + C$$

3) $\int \frac{2}{(3x+1)^5} dx$

$$= \int 2(3x+1)^{-5} dx$$

$$= 2\left(-\frac{1}{4}\right)\left(\frac{1}{3}\right)(3x+1)^{-4} + C$$

$$= -\frac{1}{6(3x+1)^4} + C$$

4) $\int_0^2 \sqrt{4x+1}\, dx$

$$= \int_0^2 (4x+1)^{1/2} dx$$

$$= \left[\frac{2}{3}\left(\frac{1}{4}\right)(4x+1)^{3/2}\right]_0^2$$

$$= \frac{1}{6}(9)^{3/2} - \frac{1}{6}(1)^{3/2}$$

$$= \frac{27}{6} - \frac{1}{6}$$

$$= \frac{26}{6} = 4\frac{1}{3}$$

Worksheet: Integration 1 **SCHOLAR: p81 Q1 - 11**

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

$$\int (ax+b)^n dx = \left(\frac{1}{a}\right)\left(\frac{1}{n+1}\right)(ax+b)^{n+1} + C$$

1.

a) $\int 8x^3 dx$ b) $\int \frac{6}{x^3} dx$ c) $\int \sqrt{x} dx$ d) $\int \frac{1}{\sqrt{x}} dx$

e) $\int (3x^2 - 4x) dx$ f) $\int \frac{3x^4 + 6}{x^2} dx$ g) $\int (x^2 + 1)^2 dx$ h) $\int \frac{x-3}{\sqrt{x}} dx$

i) $\int \frac{2x^2 - 1}{\sqrt{x}} dx$ j) $\int (x^2 + 3x)^2 dx$ k) $\int \frac{3 - x^5}{x^2} dx$ l) $\int \left(x + \frac{1}{x}\right)^2 dx$

m) $\int (2x+1)^3 dx$ n) $\int (3x-1)^4 dx$ o) $\int (1-2x)^5 dx$ p) $\int \sqrt{2x+5} dx$

q) $\int (8x+3)^3 dx$ r) $\int (5-4x)^2 dx$ s) $\int \frac{1}{\sqrt{4x+1}} dx$ t) $\int \sqrt{3x+2} dx$

u) $\int \frac{1}{(2x-3)^2} dx$ v) $\int \frac{1}{(3x+7)^3} dx$ w) $\int \frac{2}{(4x-5)^3} dx$ x) $\int \frac{1}{\sqrt{5-2x}} dx$

y) $\int (2x+1)^{\frac{1}{3}} dx$ z) $\int \frac{1}{(2x-5)^{\frac{3}{2}}} dx$

2.

a) $\int_0^2 (3+2x) dx$ b) $\int_1^2 \left(x - 2 + \frac{3}{x^2}\right) dx$ c) $\int_1^3 \left(x^2 + \frac{1}{x^2}\right) dx$ d) $\int_0^4 \sqrt{2x+1} dx$

e) $\int_1^6 \sqrt{x+3} dx$ f) $\int_0^1 (3x+1)^3 dx$ g) $\int_1^4 \frac{x^4+1}{\sqrt{x}} dx$ h) $\int_1^2 \frac{1}{(2x-1)^4} dx$

i) $\int_1^5 \frac{1}{\sqrt{2x-1}} dx$ j) $\int_1^2 \left(\frac{2}{x^2} - \frac{5}{x^4}\right) dx$ k) $\int_1^4 \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$ l) $\int_1^4 \frac{2x^2+3}{\sqrt{x}} dx$

m) $\int_0^1 \frac{1}{(4+5x)^2} dx$ n) $\int_5^{12} (x-4)^{\frac{1}{3}} dx$

ANSWERS

1.

a) $2x^4 + C$

b) $-\dfrac{3}{x^2} + C$

c) $\dfrac{2}{3}x^{\frac{3}{2}} + C$

d) $2\sqrt{x} + C$

e) $x^3 - 2x^2 + C$

f) $x^3 - \dfrac{6}{x} + C$

g) $\dfrac{1}{5}x^5 + \dfrac{2}{3}x^3 + x + C$

h) $\dfrac{2}{3}x^{\frac{3}{2}} - 6\sqrt{x} + C$

i) $\dfrac{4}{5}x^{\frac{5}{2}} - 2\sqrt{x} + C$

j) $\dfrac{1}{5}x^5 + \dfrac{3}{2}x^4 + 3x^3 + C$

k) $-\dfrac{3}{x} - \dfrac{1}{4}x^4 + C$

l) $\dfrac{1}{3}x^3 + 2x - \dfrac{1}{x} + C$

m) $\dfrac{1}{8}(2x+1)^4 + C$

n) $\dfrac{1}{15}(3x-1)^5 + C$

o) $-\dfrac{1}{12}(1-2x)^6 + C$

p) $\dfrac{1}{3}(2x+5)^{\frac{3}{2}} + C$

q) $\dfrac{1}{32}(8x+3)^4 + C$

r) $-\dfrac{1}{12}(5-4x)^3 + C$

s) $\dfrac{1}{2}\sqrt{4x+1} + C$

t) $\dfrac{2}{9}(3x+2)^{\frac{3}{2}} + C$

u) $-\dfrac{1}{2(2x-3)} + C$

v) $-\dfrac{1}{6(3x+7)^2} + C$

w) $-\dfrac{1}{4(4x-5)^2} + C$

x) $-\sqrt{5-2x} + C$

y) $\dfrac{3}{8}(2x+1)^{\frac{4}{3}} + C$

z) $-\dfrac{1}{\sqrt{2x-5}} + C$

2.

a) 10

b) 1

c) $9\dfrac{1}{3}$

d) $8\dfrac{2}{3}$

e) $12\dfrac{2}{3}$

f) $21\dfrac{1}{4}$

g) $115\dfrac{5}{9}$

h) $\dfrac{13}{81}$

i) 2

j) $-\dfrac{11}{24}$

k) $2\dfrac{2}{3}$

l) $30\dfrac{4}{5}$

m) $\dfrac{1}{36}$

n) $11\dfrac{1}{4}$

INTEGRATION OF EXPONENTIAL FUNCTIONS

Integration is the reverse process of differentiation, therefore observing the rules in differentiating exponentials, the following rules apply

$$\int e^x dx = e^x + C$$

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$$

Only applies when integrating the exponential of Linear Functions.

*Does **not** apply to* $\int e^{x^2} dx$.

Examples

1) $\int \left(6e^{3x} + 4e^{-x}\right) dx$

$= \frac{1}{3}(6)e^{3x} + 4(-1)e^{-x} + C$

$= \underline{2e^{3x} - 4e^{-x} + C}$

2) $\int \frac{8}{e^{2x}} dx$

$= \int 8e^{-2x} dx$

$= -\frac{1}{2}(8)e^{-2x} + C$

$= \underline{-4e^{-2x} + C}$

3) $\int \left(e^x + \frac{1}{e^x}\right)^2 dx$

$= \int e^{2x} + 2 + \frac{1}{e^{2x}} dx$

$= \int e^{2x} + 2 + e^{-2x} dx$

$= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C$

$= \underline{\frac{1}{2}e^{2x} + 2x - \frac{1}{2e^{2x}} + C}$

4) $\int_0^1 \frac{\left(e^{2x} - 1\right)^2}{e^x} dx$

$= \int_0^1 \frac{e^{4x} - 2e^{2x} + 1}{e^x} dx$

$= \int_0^1 \frac{e^{4x}}{e^x} - \frac{2e^{2x}}{e^x} + \frac{1}{e^x} dx$

$= \int_0^1 e^{3x} - 2e^x + e^{-x} dx$

$= \left[\frac{1}{3}e^{3x} - 2e^x - e^{-x}\right]_0^1$

$= \frac{1}{3}e^3 - 2e^1 - e^{-1} - \frac{1}{3}e^0 + 2e + e^0$

$= \underline{\frac{1}{3}e^3 - 2e - e^{-1} + \frac{8}{3}}$

Worksheet: Integration 2

$$\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$$

1.

a) $\int e^{2x}dx$ b) $\int e^{4x+1}dx$ c) $\int e^{-2x}dx$ d) $\int e^{\frac{1}{2}x}dx$

e) $\int 3e^{-x}dx$ f) $\int 6e^{3x}dx$ g) $\int 2e^{4x}dx$ h) $\int \frac{1}{2}e^{6x}dx$

i) $\int 8e^{-2x}dx$ j) $\int (2e^{2x}+x)dx$ k) $\int \left(e^{2x}+\frac{1}{e^{2x}}\right)dx$ l) $\int \left(3e^{-3x}-\frac{1}{2}e^{2x}\right)dx$

m) $\int (2e^{x}-e^{-2x})dx$ n) $\int e^{x}(e^{x}+1)dx$ o) $\int e^{2x}(e^{x}-1)dx$ p) $\int (e^{x}+1)^2 dx$

q) $\int (e^{x}+e^{-x})^2 dx$ r) $\int \frac{e^{x}+1}{e^{x}}dx$ s) $\int (8e^{-4x}+2e^{-6x})dx$ t) $\int \frac{2}{e^{4x}}dx$

2. Evaluate, expressing each answer in terms of e:

a) $\int_1^2 (e^{x}+1)dx$ b) $\int_0^1 e^{x+2}dx$ c) $\int_1^2 e^{2x}dx$ d) $\int_0^2 e^{-3x}dx$

e) $\int_0^1 e^{-x}dx$ f) $\int_0^2 e^{\frac{1}{2}x}dx$ g) $\int_2^3 \frac{1}{2}e^{-3x}dx$ h) $\int_0^1 (e^{x}+e^{-x})dx$

3. Show that $\int_0^1 (e^{x}-1)^2 dx = \frac{1}{2}(e^2-4e+5)$.

ANSWERS

1.
a) $\frac{1}{2}e^{2x}+C$ b) $\frac{1}{4}e^{4x+1}+C$ c) $-\frac{1}{2}e^{-2x}+C$ d) $2e^{\frac{1}{2}x}+C$

e) $-3e^{-x}+C$ f) $2e^{3x}+C$ g) $\frac{1}{2}e^{4x}+C$ h) $\frac{1}{12}e^{6x}+C$

i) $-4e^{-2x}+C$ j) $e^{2x}+\frac{1}{2}x^2+C$ k) $\frac{1}{2}e^{2x}-\frac{1}{2}e^{-2x}+C$ l) $-e^{-3x}-\frac{1}{4}e^{2x}+C$

m) $2e^{x}+\frac{1}{2}e^{-2x}+C$ n) $\frac{1}{2}e^{2x}+e^{x}+C$ o) $\frac{1}{3}e^{3x}-\frac{1}{2}e^{2x}+C$ p) $\frac{1}{2}e^{2x}+2e^{x}+x+C$

q) $\frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}+C$ r) $x-e^{-x}+C$ s) $-2e^{-4x}-\frac{1}{3}e^{-6x}+C$ t) $-\frac{1}{2}e^{-4x}+C$

2.
a) e^2-e+1 b) e^3-e^2 c) $\frac{1}{2}e^4-\frac{1}{2}e^2$ d) $\frac{1}{3}-\frac{1}{3}e^{-6}$

e) $1-e^{-1}$ f) $2e-2$ g) $\frac{1}{6}e^{-6}-\frac{1}{6}e^{-9}$ h) $e-e^{-1}$

3. Proof.

INTEGRATION OF NATURAL LOG (LN)

Ln functions do not exist for NEGATIVE values.

The magnitude of x is used to ensure that the x value is positive.

$$|x| = x \quad \text{if} \quad x \geq 0$$
$$|x| = -x \quad \text{if} \quad x < 0$$

$$\boxed{\begin{aligned} &\int \frac{1}{x}\,dx = \ln|x| + C \\ &\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C \end{aligned}}$$

Applies to all non-zero values of x.

Magnitude signs not necessary if x is positive.

Only applies to linear integrals

eg. not $\int \frac{1}{x^2+1}\,dx.$

Examples

1) $\int \frac{8}{2x+1}\,dx.$

$= 8\left(\frac{1}{2}\right)\ln|2x+1| + C$

$= \underline{4\ln|2x+1| + C}$

2) $\int \left(\frac{2}{x} + \frac{4}{6x+1} - \frac{6}{1-3x}\right)dx.$

$= 2\ln x + 4\left(\frac{1}{6}\right)\ln|6x+1| - 6\left(-\frac{1}{3}\right)\ln|1-3x| + C$

$= \underline{2\ln x + \frac{2}{3}\ln|6x+1| + 2\ln|1-3x| + C}$

3) Show that $\int_2^{14} \frac{1}{2x-1}\,dx = \ln 3$

LHS

$= \left[\frac{1}{2}\ln|2x-1|\right]_2^{14}$

$= \frac{1}{2}\ln 27 - \frac{1}{2}\ln 3$

$= \frac{1}{2}\ln\left(\frac{27}{3}\right)$

$= \frac{1}{2}\ln 9$

$= \ln 9^{\frac{1}{2}} = \underline{\ln 3}$

Worksheet: Integration 3

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b)+C$$

1.

a) $\int \frac{1}{x+4}\,dx$ b) $\int \frac{1}{2x+1}\,dx$ c) $\int \frac{1}{4x+3}\,dx$ d) $\int \frac{1}{3x-2}\,dx$

e) $\int \frac{3}{x}\,dx$ f) $\int \frac{1}{1-x}\,dx$ g) $\int \frac{8}{2x+3}\,dx$ h) $\int \frac{3}{3x+1}\,dx$

i) $\int \frac{2}{1-4x}\,dx$ j) $\int \frac{10}{5x+1}\,dx$ k) $\int \frac{1}{3-2x}\,dx$ l) $\int \frac{1}{2x}\,dx$

m) $\int \frac{6}{2x-1}\,dx$ n) $\int \frac{6}{1-3x}\,dx$ o) $\int \frac{6}{4x+1}\,dx$ p) $\int \left(\frac{1}{2x+5}+\frac{1}{x-3}\right)dx$

q) $\int \left(\frac{2}{3x+1}+\frac{1}{2x-1}-\frac{6}{4x+3}\right)dx$

2. Evaluate, giving each answer correct to 3 decimal places:

a) $\int_3^7 \frac{1}{x}\,dx$ b) $\int_1^3 \frac{1}{2x-1}\,dx$ c) $\int_2^5 \frac{1}{x+1}\,dx$ d) $\int_0^1 \frac{1}{3x+5}\,dx$

e) $\int_1^4 \frac{2}{2x+3}\,dx$ f) $\int_4^{10} \frac{5}{x-3}\,dx$ g) $\int_2^5 \frac{6}{2x-3}\,dx$ h) $\int_4^5 \frac{2}{x-3}\,dx$

i) $\int_2^6 \frac{4}{x}\,dx$ j) $\int_5^9 \frac{1}{x-3}\,dx$ k) $\int_0^1 \frac{1}{3x+2}\,dx$ l) $\int_{-4}^0 \frac{1}{1-2x}\,dx$

m) $\int_0^1 \frac{8}{4x+3}\,dx$

3. Show that a) $\int_3^5 \frac{1}{x-1}\,dx = \ln 2$ b) $\int_1^{13} \frac{1}{2x+1}\,dx = \ln 3$

ANSWERS

1.

a) $\ln(x+4)+C$

b) $\frac{1}{2}\ln(2x+1)+C$

c) $\frac{1}{4}\ln(4x+3)+C$

d) $\frac{1}{3}\ln(3x-2)+C$

e) $3\ln x+C$

f) $-\ln(1-x)+C$

g) $4\ln(2x+3)+C$

h) $\ln(3x+1)+C$

i) $-\frac{1}{2}\ln(1-4x)+C$

j) $2\ln(5x+1)+C$

k) $-\frac{1}{2}\ln(3-2x)+C$

l) $\frac{1}{2}\ln x+C$

m) $3\ln(2x-1)+C$

n) $-2\ln(1-3x)+C$

o) $\frac{3}{2}\ln(4x+1)+C$

p)

q) $\frac{2}{3}\ln(3x+1)+\frac{1}{2}\ln(2x-1)-\frac{3}{2}\ln(4x+3)+C$

2.

a) 0.847

b) 0.805

c) 0.693

d) 0.157

e) 0.788

f) 9.730

g) 5.838

h) 1.386

i) 4.394

j) 1.099

k) 0.305

l) 1.099

m) 1.695

3. Proof.

INTEGRATION OF TRIG FUNCTIONS

$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$
$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$
$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$
$$\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + C$$

Examples

1) $\int (3\cos x - 4\sin 2x)\,dx$

$= 3\sin x - 4\left(\frac{1}{2}\right)(-\cos 2x) + C$

$= \underline{3\sin x + 2\cos 2x + C}$

2) $\int 6\sec^2 4x\,dx$

$= 6\left(\frac{1}{4}\right)\tan 4x + C$

$= \underline{\frac{3}{2}\tan 4x + C}$

3) Show that $\int_0^{\frac{\pi}{6}} (\sin 2x + \sec^2 x)\,dx = \frac{4\sqrt{3}+3}{12}$

$\int_0^{\frac{\pi}{6}} (\sin 2x + \sec^2 x)\,dx$

$= \left[-\frac{1}{2}\cos 2x + \tan x\right]_0^{\frac{\pi}{6}}$

$= -\frac{1}{2}\cos\frac{\pi}{3} + \tan\frac{\pi}{6} + \frac{1}{2}\cos 0 - \tan 0$

$= -\frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{\sqrt{3}} + \frac{1}{2} - 0$

$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{\sqrt{3}}$

$= \frac{1}{4} + \frac{1}{\sqrt{3}} = \frac{\sqrt{3}+4}{4\sqrt{3}}$

$= \frac{\sqrt{3}+4}{4\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)$

$= \underline{\frac{3+4\sqrt{3}}{12}}$

Worksheet: Integration 4 **SCHOLAR: p85 Q12 - 27**

$$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)+C$$

$$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+C$$

$$\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)+C$$

1.
a) $\int \cos 2x dx$ b) $\int \sin 3x dx$ c) $\int \sec^2 2x dx$ d) $\int 6\cos 3x dx$

e) $\int 2\sin 4x dx$ f) $\int \sec^2 3x dx$ g) $\int 4\cos 6x dx$ h) $\int 2\sin 2x dx$

i) $\int \sec^2 4x dx$ j) $\int (\sec^2 x+1)dx$ k) $\int (3\cos 3x+2\sin x)dx$ l) $\int (8\cos 2x-4\sin 2x)dx$

m) $\int (6\sin 2x+2\cos 4x)dx$ n) $\int 6\sec^2 2x dx$ o) $\int \sec^2 \frac{1}{2}x dx$

2.
a) $\int_0^{\frac{\pi}{2}} \cos x dx$ b) $\int_0^{\frac{\pi}{3}} \sin x dx$ c) $\int_0^{\frac{\pi}{4}} \cos 2x dx$

d) $\int_0^{\frac{\pi}{2}} \sin 2x dx$ e) $\int_0^{\frac{\pi}{4}} \sec^2 x dx$ f) $\int_0^{\frac{\pi}{4}} (\sin x+\cos x)dx$

g) $\int_0^{\frac{\pi}{8}} \cos 4x dx$

3. Show that $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x dx = \frac{2\sqrt{3}}{3}$.

ANSWERS

1.

a) $\frac{1}{2}\sin 2x + C$ b) $-\frac{1}{3}\cos 3x + C$ c) $\frac{1}{2}\tan 2x + C$ d) $2\sin 3x + C$

e) $-\frac{1}{2}\cos 4x + C$ f) $\frac{1}{3}\tan 3x + C$ g) $\frac{2}{3}\sin 6x + C$ h) $-\cos 2x + C$

i) $\frac{1}{4}\tan 4x + C$ j) $\tan x + x + C$ k) $\sin 3x - 2\cos x + C$ l) $4\sin 2x + 2\cos 2x + C$

m) $-3\cos 2x + \frac{1}{2}\sin 4x + C$ n) $3\tan 2x + C$ o) $2\tan\frac{1}{2}x + C$

2.

a) 1 b) $\frac{1}{2}$ c) $\frac{1}{2}$

d) 1 e) 1 f) 1

g) $\frac{1}{4}$

3. Proof

INTEGRATION BY SUBSTITUTION

When the derivative of one part of the integrand is related to another part of the integrand, Integration by substitution can be used.

NOTE***: Look for the relationship of one part being the derivative of the other.***

Examples

1) $\int x\left(x^2+1\right)^5 dx$ using $u = x^2+1$.

Let $u = x^2+1$

$\frac{du}{dx} = 2x$

$du = 2xdx$

$\frac{1}{2}du = xdx$

$$\int x\left(x^2+1\right)^5 dx$$

$$= \int \frac{1}{2}u^5 du$$

$$= \frac{1}{2}\left(\frac{1}{6}\right)u^6 + C$$

$$= \underline{\frac{1}{12}\left(x^2+1\right)^6 + C}$$

2) $\int x^2\sqrt{x^3+4}\,dx$ using $u = \left(x^3+4\right)$.

Let $u = x^3+4$

$\frac{du}{dx} = 3x^2$

$du = 3x^2dx$

$\frac{1}{3}du = x^2dx$

$$\int x^2\sqrt{x^3+4}\,dx$$

$$= \int \frac{1}{3}u^{1/2}du$$

$$= \frac{1}{3}\left(\frac{2}{3}\right)u^{3/2} + C$$

$$= \underline{\frac{2}{9}\left(x^3+4\right)^{3/2} + C}$$

3) $\int \sin^2 x\cos x\,dx$ using $u = \sin x$.

Let $u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\,dx$

$$\int \sin^2 x\cos x\,dx$$

$$= \int u^2 du$$

$$= \frac{1}{3}u^3 + C$$

$$= \underline{\frac{1}{3}\sin^3 x + C}$$

Worksheet: Integration by Substitution 1 **SCHOLAR: p87 Q29 - 37**

1. Use the suggested substitution given in brackets to find:

a) $\int x(x^2+1)^3\,dx \quad (u=x^2+1)$

b) $\int x^2(x^3+1)^5\,dx \quad (u=x^3+1)$

c) $\int x(x^2+3)^7\,dx \quad (u=x^2+3)$

d) $\int x^3(x^4+1)^9\,dx \quad (u=x^4+1)$

e) $\int x(2x^2+1)^4\,dx \quad (u=2x^2+1)$

f) $\int x(1-x^2)^6\,dx \quad (u=1-x^2)$

g) $\int x^2(2x^3-1)^5\,dx \quad (u=2x^3-1)$

h) $\int xe^{x^2}\,dx \quad (u=x^2)$

i) $\int \cos x e^{\sin x}\,dx \quad (u=\sin x)$

j) $\int \sin^2 x\cos x\,dx \quad (u=\sin x)$

k) $\int \cos^3 x\sin x\,dx \quad (u=\cos x)$

l) $\int x\sin(x^2)\,dx \quad (u=x^2)$

m) $\int x\sqrt{x^2+1}\,dx \quad (u=x^2+1)$

n) $\int \sec^2 x\tan^4 x\,dx \quad (u=\tan x)$

o) $\int x(2x^2+1)^3\,dx \quad (u=2x^2+1)$

p) $\int x^2e^{x^3}\,dx \quad (u=x^3)$

q) $\int \sin^4 x\cos x\,dx \quad (u=\sin x)$

r) $\int x^2(2x^3+1)^4\,dx \quad (u=2x^3+1)$

s) $\int x^2\sqrt{x^3+1}\,dx \quad (u=x^3+1)$

t) $\int \cos x\sqrt{1+\sin x}\,dx \quad (u=1+\sin x)$

u) $\int \frac{\ln x}{x}\,dx \quad (u=\ln x)$

v) $\int x(x^2-3)^5\,dx \quad (u=x^2-3)$

w) $\int x\sqrt{1-x^2}\,dx \quad (u=1-x^2)$

x) $\int x^2(x^3-1)^2\,dx \quad (u=x^3-1)$

y) $\int xe^{2x^2}\,dx \quad (u=2x^2)$

z) $\int 8\cos x\sin^3 x\,dx \quad (u=\sin x)$

ANSWERS

1.

a) $\frac{1}{8}(x^2+1)^4+C$

b) $\frac{1}{18}(x^3+1)^6+C$

c) $\frac{1}{16}(x^2+3)^8+C$

d) $\frac{1}{40}(x^4+1)^{10}+C$

e) $\frac{1}{20}(2x^2+1)^5+C$

f) $-\frac{1}{14}(1-x^2)^7+C$

g) $\frac{1}{36}(2x^3-1)^6+C$

h) $\frac{1}{2}e^{x^2}+C$

i) $e^{\sin x}+C$

j) $\frac{1}{3}\sin^3 x+C$

k) $-\frac{1}{4}\cos^4 x+C$

l) $-\frac{1}{2}\cos(x^2)+C$

m) $\frac{1}{3}(x^2+1)^{\frac{3}{2}}+C$

n) $\frac{1}{5}\tan^5 x+C$

o) $\frac{1}{16}(2x^2+1)^4+C$

p) $\frac{1}{3}e^{x^3}+C$

q) $\frac{1}{5}\sin^5 x+C$

r) $\frac{1}{30}(2x^3+1)^5+C$

s) $\frac{2}{9}(x^3+1)^{\frac{3}{2}}+C$

t) $\frac{2}{3}(1+\sin x)^{\frac{3}{2}}+C$

u) $\frac{1}{2}(\ln x)^2+C$

v) $\frac{1}{12}(x^2-3)^6+C$

w) $-\frac{1}{3}(1-x^2)^{\frac{3}{2}}+C$

x) $\frac{1}{9}(x^3-1)^3+C$

y) $\frac{1}{4}e^{2x^2}+C$

z) $2\sin^4 x+C$

INTEGRATION BY SUBSTITUTION 2

Examples

4) $\int \frac{x}{3x^2+1}\,dx$ using $u = 3x^2+1$

$u = 3x^2+1$

$\frac{du}{dx} = 6x$

$\frac{1}{6}du = xdx$

$$\int \frac{x}{3x^2+1}\,dx$$

$$= \int \frac{1}{6}\left(\frac{1}{u}\right)du$$

$$= \frac{1}{6}\ln u + C$$

$$= \underline{\frac{1}{6}\ln\left|3x^2+1\right| + C}$$

5) Use $u = 1+\cos 2x$ to find $\int \frac{\sin 2x}{\sqrt{1+\cos 2x}}\,dx$.

$u = 1+\cos 2x$

$du = -2\sin 2x dx$

$-\frac{1}{2}du = \sin 2x dx$

$$\int \frac{\sin 2x}{\sqrt{1+\cos 2x}}\,dx$$

$$= \int -\frac{1}{2}u^{-1/2}du$$

$$= -\frac{1}{2}(2)u^{1/2} + C$$

$$= -\sqrt{1+\cos 2x} + C$$

$$= \underline{C - \sqrt{1+\cos 2x}}$$

Worksheet: Integration by Substitution 2

1. Use the suggested substitution given in brackets to find:

a) $\int \frac{x}{x^2+1}\,dx \quad (u = x^2+1)$

b) $\int \frac{x^2}{x^3+1}\,dx \quad (u = x^3+1)$

c) $\int \frac{e^x}{e^x+1}\,dx \quad (u = e^x+1)$

d) $\int \frac{x^3}{1+x^4}\,dx \quad (u = 1+x^4)$

e) $\int \frac{e^{2x}}{e^{2x}-1}\,dx \quad (u = e^{2x}-1)$

f) $\int \frac{x}{2x^2+1}\,dx \quad (u = 2x^2+1)$

g) $\int \frac{\sec^2 x}{\tan x}\,dx \quad (u = \tan x)$

h) $\int \frac{x}{\sqrt{x^2+4}}\,dx \quad (u = x^2+4)$

i) $\int \frac{x^2}{\sqrt{x^3+1}}\,dx \quad (u = x^3+1)$

j) $\int \frac{x}{\sqrt{3x^2-1}}\,dx \quad (u = 3x^2-1)$

k) $\int \frac{e^{2x}}{\sqrt{e^{2x}+1}}\,dx \quad (u = e^{2x}+1)$

l) $\int \frac{x}{\sqrt{2x^2+3}}\,dx \quad (u = 2x^2+3)$

m) $\int \frac{\cos x}{\sqrt{\sin x}}\,dx \quad (u = \sin x)$

n) $\int \frac{\sin 2x}{\sqrt{\cos 2x}}\,dx \quad (u = \cos 2x)$

o) $\int \frac{x}{(x^2+1)^4}\,dx \quad (u = x^2+1)$

p) $\int \frac{x^2}{(x^3+1)^2}\,dx \quad (u = x^3+1)$

q) $\int \frac{x}{(1-x^2)^3}\,dx \quad (u = 1-x^2)$

r) $\int \frac{\cos x}{\sin^6 x}\,dx \quad (u = \sin x)$

s) $\int \frac{\sin x}{\cos^4 x}\,dx \quad (u = \cos x)$

t) $\int \frac{\cos x}{1+\sin x}\,dx \quad (u = 1+\sin x)$

u) $\int \frac{\sin x}{1+\cos x}\,dx \quad (u = 1+\cos x)$

v) $\int \frac{\cos 2x}{1+\sin 2x}\,dx \quad (u = 1+\sin 2x)$

w) $\int \frac{\sin 4x}{1-\cos 4x}\,dx \quad (u = 1-\cos 4x)$

x) $\int \frac{4x}{x^2+1}\,dx \quad (u = x^2+1)$

y) $\int \frac{e^{3x}}{\sqrt{1-e^{3x}}}\,dx \quad (u = 1-e^{3x})$

z) $\int \frac{\cos 2x}{(1+\sin 2x)^4}\,dx \quad (u = 1+\sin 2x)$

2 a) By writing $\tan x$ as $\frac{\sin x}{\cos x}$ and using the substitution $u = \cos x$, find $\int \tan x\,dx$.

b) Use a similar method to that in part a) to find $\int \cot x\,dx$.

ANSWERS

1.

a) $\frac{1}{2}\ln(x^2+1)+C$

b) $\frac{1}{3}\ln(x^3+1)+C$

c) $\ln(e^x+1)+C$

d) $\frac{1}{4}\ln(1+x^4)+C$

e) $\frac{1}{2}\ln(e^{2x}-1)+C$

f) $\frac{1}{4}\ln(2x^2+1)+C$

g) $\ln(\tan x)+C$

h) $\sqrt{x^2+4}+C$

i) $\frac{2}{3}\sqrt{x^3+1}+C$

j) $\frac{1}{3}\sqrt{3x^2-1}+C$

k) $\sqrt{e^{2x}+1}+C$

l) $\frac{1}{2}\sqrt{2x^2+3}+C$

m) $2\sqrt{\sin x}+C$

n) $-\sqrt{\cos 2x}+C$

o) $-\frac{1}{6(x^2+1)^3}+C$

p) $-\frac{1}{3(x^3+1)}+C$

q) $\frac{1}{4(1-x^2)^2}+C$

r) $-\frac{1}{5\sin^5 x}+C$

s) $\frac{1}{3\cos^3 x}+C$

t) $\ln(1+\sin x)+C$

u) $-\ln(1+\cos x)+C$

v) $\frac{1}{2}\ln(1+\sin 2x)+C$

w) $\frac{1}{4}\ln(1-\cos 4x)+C$

x) $2\ln(x^2+1)+C$

y) $-\frac{2}{3}\sqrt{1-e^{3x}}+C$

z) $-\frac{1}{6(1+\sin 2x)^3}+C$

2 a) $-\ln(\cos x)+C$

b) $\ln(\sin x)+C$

Examples **INTEGRATION BY SUBSTITUTION 3**

6) If $u = x+1$ find $\int x(x+1)^3\,dx$.

$u = x+1$
$du = dx$
$x = u-1$

$$\int x(x+1)^3\,dx$$
$$= \int u^3(u-1)\,du$$
$$= \int u^4 - u^3\,du$$
$$= \frac{1}{5}u^5 - \frac{1}{4}u^4 + C$$
$$= \underline{\frac{1}{5}(x+1)^5 - \frac{1}{4}(x+1)^4 + C}$$

7) If $u = x-2$ find $\int \frac{x+1}{\sqrt{x-2}}\,dx$.

$u = x-2$
$du = dx$
$x = u+2$
$x+1 = u+3$

$$\int \frac{x+1}{\sqrt{x-2}}\,dx$$
$$= \int u^{-1/2}(u+3)\,du$$
$$= \int u^{1/2} + 3u^{-1/2}\,du$$
$$= \frac{2}{3}u^{3/2} + 6u^{1/2} + C$$
$$= \underline{\frac{2}{3}(x-2)^{3/2} + 6\sqrt{x-2} + C}$$

8) Use $u = x^2+1$ to find $\int x^3\sqrt{x^2+1}\,dx$.

$u = x^2+1$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$
$x^2 = u-1$

$$\int x^3\sqrt{x^2+1}\,dx$$
$$= \int \frac{1}{2}(u-1)u^{1/2}\,du$$
$$= \int \frac{1}{2}\left(u^{3/2} - u^{1/2}\right)du$$
$$= \frac{1}{2}\left(\frac{2}{5}\right)u^{5/2} - \frac{1}{2}\left(\frac{2}{3}\right)u^{3/2} + C$$
$$= \underline{\frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C}$$

Worksheet: Integration by Substitution 3

1. Use the suggested substitution given in brackets to find:

a) $\int x(x+1)^5\,dx \quad (u=x+1)$

b) $\int x(x-1)^3\,dx \quad (u=x-1)$

c) $\int x(x+2)^4\,dx \quad (u=x+2)$

d) $\int x(x-4)^7\,dx \quad (u=x-4)$

e) $\int (x+1)(x+2)^3\,dx \quad (u=x+2)$

f) $\int (x-1)(x+1)^6\,dx \quad (u=x+1)$

g) $\int (x+2)(x-1)^4\,dx \quad (u=x-1)$

h) $\int (x+1)(x+3)^5\,dx \quad (u=x+3)$

i) $\int x\sqrt{x+1}\,dx \quad (u=x+1)$

j) $\int x\sqrt{x-2}\,dx \quad (u=x-2)$

k) $\int \frac{x}{\sqrt{x+1}}\,dx \quad (u=x+1)$

l) $\int \frac{x}{\sqrt{x-4}}\,dx \quad (u=x-4)$

m) $\int \frac{x+1}{\sqrt{x+2}}\,dx \quad (u=x+2)$

n) $\int \frac{x+1}{\sqrt{x-1}}\,dx \quad (u=x-1)$

o) $\int x^3(x^2+1)^4\,dx \quad (u=x^2+1)$

p) $\int x^3(x^2-1)^3\,dx \quad (u=x^2-1)$

q) $\int \sin^4 x\cos^3 x\,dx \quad (u=\sin x)$

r) $\int \cos^2 x\sin^3 x\,dx \quad (u=\cos x)$

s) $\int \frac{x}{x+1}\,dx \quad (u=x+1)$

t) $\int \frac{x}{x-2}\,dx \quad (u=x-2)$

u) $\int \frac{x}{(x+1)^2}\,dx \quad (u=x+1)$

v) $\int x(2x+1)^3\,dx \quad (u=2x+1)$

w) $\int x(2x-1)^5\,dx \quad (u=2x-1)$

x) $\int x(3x+1)^4\,dx \quad (u=3x+1)$

y) $\int x^2\sqrt{x+1}\,dx \quad (u=x+1)$

z) $\int \frac{x^2}{\sqrt{x-1}}\,dx \quad (u=x-1)$

2. Make use of the substitution $u = \sec x$ to find $\int \sec^3 x\tan x\,dx$.

3. Make use of the substitution u = sin x to find $\int \cos^3 x\,dx$.

4. Make use of the substitution u = cos x to find $\int \sin^3 x\,dx$.

5. Make use of the substitution u = $x^2 + 1$ to find $\int \frac{x^3}{\sqrt{x^2+1}}\,dx$.

ANSWERS

1.

a) $\frac{1}{7}(x+1)^7-\frac{1}{6}(x+1)^6+C$

b) $\frac{1}{5}(x-1)^5+\frac{1}{4}(x-1)^4+C$

c) $\frac{1}{6}(x+2)^6-\frac{2}{5}(x+2)^5+C$

d) $\frac{1}{9}(x-4)^9+\frac{1}{2}(x-4)^8+C$

e) $\frac{1}{5}(x+2)^5-\frac{1}{4}(x+2)^4+C$

f) $\frac{1}{8}(x+1)^8-\frac{2}{7}(x+1)^7+C$

g) $\frac{1}{6}(x-1)^6+\frac{3}{5}(x-1)^5+C$

h) $\frac{1}{7}(x+3)^7-\frac{1}{3}(x+3)^6+C$

i) $\frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{2}{3}(x+1)^{\frac{3}{2}}+C$

j) $\frac{2}{5}(x-2)^{\frac{5}{2}}+\frac{4}{3}(x-2)^{\frac{3}{2}}+C$

k) $\frac{2}{3}(x+1)^{\frac{3}{2}}-2\sqrt{x+1}+C$

l) $\frac{2}{3}(x-4)^{\frac{3}{2}}-8\sqrt{x-4}+C$

m) $\frac{2}{3}(x+2)^{\frac{3}{2}}-2\sqrt{x+2}+C$

n) $\frac{2}{3}(x-1)^{\frac{3}{2}}+4\sqrt{x-1}+C$

o) $\frac{1}{12}(x^2+1)^6-\frac{1}{10}(x^2+1)^5+C$

p) $\frac{1}{10}(x^2-1)^5+\frac{1}{8}(x^2-1)^4+C$

q) $\frac{1}{5}\sin^5 x-\frac{1}{7}\sin^7 x+C$

r) $-\frac{1}{3}\cos^3 x+\frac{1}{5}\cos^5 x+C$

s) $x+1-\ln(x+1)+C$

t) $x-2+2\ln(x-2)+C$

u) $\ln(x+1)+\frac{1}{x+1}+C$

v) $\frac{1}{20}(2x+1)^5-\frac{1}{16}(2x+1)^4+C$

w) $\frac{1}{28}(2x-1)^7+\frac{1}{24}(2x-1)^6+C$

x) $\frac{1}{54}(3x+1)^6-\frac{1}{45}(3x+1)^5+C$

y) $\frac{2}{7}(x+1)^{\frac{7}{2}}-\frac{4}{5}(x+1)^{\frac{5}{2}}+\frac{2}{3}(x+1)^{\frac{3}{2}}C$

z) $\frac{2}{5}(x-1)^{\frac{5}{2}}+\frac{4}{3}(x-1)^{\frac{3}{2}}+2\sqrt{x-1}+C$

2. $\frac{1}{3}\sec^3 x+C$

3. $\sin x-\frac{1}{3}\sin^3 x+C$

4. $-\cos x+\frac{1}{3}\cos^3 x+C$

5. $\frac{1}{3}(x^2+1)^{\frac{3}{2}}-\sqrt{x^2+1}+C$

INTEGRATION BY SUBSTITUTION 4

NOTE: When we use substitution to create a definite integral the limit MUST be expressed **in terms of the new variable**.

Examples

9) Use $u = x^4 + 1$ to find $\int_0^1 x^3\left(x^4+1\right)^3 dx.$

$$\int_1^2 \frac{1}{4}u^3 du$$

$$= \left[\frac{1}{4}\left(\frac{1}{4}\right)u^4\right]_1^2 = \left[\frac{1}{16}u^4\right]_1^2$$

$$= \frac{1}{16}(16) - \frac{1}{16}(1)$$

$$= 1 - \frac{1}{16} = \underline{\frac{15}{16}}$$

$u = x^4 + 1$

$du = 4x^3 dx$

$\frac{1}{4}du = x^3 dx$

Limits

$x = 0 \quad u = 1$

$x = 1 \quad u = 2$

10) Use $u = \cos x$ to find $\int_0^{\frac{\pi}{3}} \tan x dx.$

$$\int_1^{0.5} -\frac{1}{u} du$$

$$= \left[-\ln|u|\right]_1^{0.5}$$

$$= -\ln 0.5 + \ln 1$$

$$= \ln 1 - \ln 0.5$$

$$= \ln\left(\frac{1}{0.5}\right)$$

$$= \underline{\ln 2}$$

$u = \cos x$

$du = -\sin x dx$

$-1du = \sin x dx$

$\tan x = \frac{\sin x}{\cos x}$

$\tan x = \frac{-1}{u} du$

Limits

$x = 0 \quad u = 1$

$x = \frac{\pi}{3} \quad u = 2$

INTEGRATION BY SUBSTITUTION 4

11) Use $u = x - 1$ to find $\int_2^5 x\sqrt{x-1}\,dx.$

$$\int_1^4 (u+1)u^{\frac{1}{2}}du$$
$$= \int_1^4 u^{\frac{3}{2}} + u^{\frac{1}{2}}du$$
$$= \left[\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right]_1^4$$
$$= \frac{2}{5}(32) + \frac{2}{3}(8) - \frac{2}{5}(1) - \frac{2}{3}(1)$$
$$= \frac{64}{5} + \frac{16}{3} - \frac{2}{5} - \frac{2}{3}$$
$$= \frac{192 + 80 - 6 - 10}{15} = \frac{256}{15} = 17\frac{1}{15}$$

$u = x - 1$
$du = dx$
$x = u + 1$

Limits
$x = 2 \quad u = 1$
$x = 5 \quad u = 4$

12) Use $u = \sin x$ to find $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx.$

$$\int_0^{\frac{\pi}{2}} \cos^3 x\,dx = \int_0^{\frac{\pi}{2}} \cos^2 x \cos x\,dx$$
$$= \int_0^1 (1 - u^2)du$$
$$= \left[u - \frac{1}{3}u^3\right]_0^1$$
$$= 1 - \frac{1}{3} - (0) = \underline{\frac{2}{3}}$$

$u = \sin x$
$du = \cos x\,dx$
$\sin^2 x + \cos^2 x = 1$
$\cos^2 x = 1 - \sin^2 x$
$= 1 - u^2$

Limits
$x = 0 \quad u = 0$
$x = \frac{\pi}{2} \quad u = 1$

Worksheet: Integration by Substitution 4 **SCHOLAR: p89 Q38 - 45**

1. Use the suggested substitution given in brackets to evaluate:

a) $\int_0^1 x(x^2+1)^3\,dx \quad (u = x^2+1)$

b) $\int_0^1 x^2(x^3+1)^4\,dx \quad (u = x^3+1)$

c) $\int_1^2 x(x^2-1)^5\,dx \quad (u = x^2-1)$

d) $\int_0^1 x^3(x^4+1)^2\,dx \quad (u = x^4+1)$

e) $\int_0^{\frac{\pi}{2}} \sin^3 x\cos x\,dx \quad (u = \sin x)$

f) $\int_0^{\pi} \sin x\cos^4 x\,dx \quad (u = \cos x)$

g) $\int_0^1 \frac{x}{x^2+1}\,dx \quad (u = x^2+1)$

h) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x}\,dx \quad (u = 1+\sin x)$

i) $\int_0^3 \frac{x}{\sqrt{x^2+16}}\,dx \quad (u = x^2+16)$

j) $\int_0^1 x(x+1)^3\,dx \quad (u = x+1)$

k) $\int_0^1 x(x+2)^4\,dx \quad (u = x+2)$

l) $\int_0^8 x\sqrt{x+1}\,dx \quad (u = x+1)$

m) $\int_0^{\frac{\pi}{6}} \sin^4 x\cos x\,dx \quad (u = \sin x)$

n) $\int_1^6 \frac{x}{\sqrt{x+3}}\,dx \quad (u = x+3)$

o) $\int_0^3 \frac{x}{\sqrt{25-x^2}}\,dx \quad (u = 25-x^2)$

p) $\int_3^4 x^3(x^2-8)^{\frac{1}{2}}\,dx \quad (u = x^2-8)$

q) $\int_0^8 \frac{x+2}{\sqrt{x+1}}\,dx \quad (u = x+1)$

2. Make use of the substitution $u = e^x - 1$ to show that $\int_1^2 \frac{e^x}{e^x-1}\,dx = \ln(e+1)$.

ANSWERS

1.

a) $1\frac{7}{8}$

b) $2\frac{1}{15}$

c) $60\frac{3}{4}$

d) $\frac{7}{12}$

e) $\frac{1}{4}$

f) $\frac{2}{5}$

g) $\frac{1}{2}\ln 2 \cong 0.347$

h) $\ln 2 \cong 0.693$

i) 1

j) $2\frac{9}{20}$

k) $26\frac{13}{30}$

l) $79\frac{7}{15}$

m) $\frac{1}{160}$

n) $6\frac{2}{3}$

o) 1

p) $74\frac{3}{14}$

q) $21\frac{1}{3}$

2. Proof.

INTEGRATION BY SUBSTITUTION 5

If the integrand contains a term in the form $\sqrt{a^2-x^2}$, it is often easier to use a trig term.

$$\boxed{\sqrt{a^2-x^2} \Rightarrow x = a\sin\theta}$$

Examples

13) Use $x = 2\sin\theta$ to find $\displaystyle\int_0^1 \frac{x+1}{\sqrt{4-x^2}}\,dx.$

$$\int_0^{\frac{\pi}{6}} \frac{(2\sin\theta+1)}{\sqrt{4-4\sin^2\theta}}(2\cos\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{(2\sin\theta+1)(2\cos\theta)}{\sqrt{4\cos^2\theta}}\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{(2\sin\theta+1)(2\cos\theta)}{2\cos\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} 2\sin\theta + 1\,d\theta$$

$$= \left[-2\cos\theta + \theta\right]_0^{\frac{\pi}{6}}$$

$$= -2\cos\frac{\pi}{6} + \frac{\pi}{6} + 2\cos 0 - 0$$

$$= -2\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6} + 2$$

$$= -\sqrt{3} + \frac{\pi}{6} + 2 = \underline{2 + \frac{\pi}{6} - \sqrt{3}}$$

$x = 2\sin\theta$

$x^2 = 4\sin^2\theta$

$dx = 2\cos\theta\, d\theta$

$\cos^2\theta = 1 - \sin^2\theta$

$4\cos^2\theta = 4 - 4\sin^2\theta$

Limits

$x = 0 \quad 2\sin\theta = 0$

$\sin\theta = 0$

$\theta = 0$

$x = 1 \quad 2\sin\theta = 1$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

Worksheet: Integration by Substitution 5 **SCHOLAR: p91 Q46 - 53**

1. Use the suggested substitution given in brackets to evaluate:

a) $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}dx \quad (x = \sin\theta)$

b) $\int_0^{1} \frac{1}{\sqrt{4-x^2}}dx \quad (x = 2\sin\theta)$

c) $\int_0^{2} \frac{2}{\sqrt{16-x^2}}dx \quad (x = 4\sin\theta)$

d) $\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}dx \quad (x = \sin\theta)$

e) $\int_0^{\frac{3}{2}} \frac{1}{\sqrt{9-x^2}}dx \quad (x = 3\sin\theta)$

f) $\int_0^{\frac{1}{2}} \frac{x+1}{\sqrt{1-x^2}}dx \quad (x = \sin\theta)$

g) $\int_0^{1} \frac{x+2}{\sqrt{4-x^2}}dx \quad (x = 2\sin\theta)$

h) $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}dx \quad (x = 2\sin\theta)$

i) $\int_0^{\frac{1}{6}} \frac{1}{\sqrt{1-9x^2}}dx \quad (x = \frac{1}{3}\sin\theta)$

j) $\int_0^{2} \frac{x+1}{\sqrt{16-x^2}}dx \quad (x = 4\sin\theta)$

2. Make use of the substitution $x = 1-\sin\theta$ to evaluate $\int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{2x-x^2}}dx.$

ANSWERS

1.

a) $\frac{\pi}{6}$

b) $\frac{\pi}{6}$

c) $\frac{\pi}{3}$

d) $1-\frac{\sqrt{3}}{2}$

e) $\frac{\pi}{6}$

f) $1-\frac{\sqrt{3}}{2}+\frac{\pi}{6}$

g) $2-\sqrt{3}+\frac{\pi}{3}$

h) $\frac{\pi}{12}$

i) $\frac{\pi}{18}$

j) $4-2\sqrt{3}+\frac{\pi}{6}$

2. $\frac{\pi}{6}$

INTEGRATION BY SUBSTITUTION 6 – Select own Substitute

Examples

14) $\int x\sqrt{x^2+1}\,dx$

$$\int \frac{1}{2}u^{\frac{1}{2}}du$$
$$=\frac{1}{2}\left(\frac{2}{3}\right)u^{\frac{3}{2}}+C$$
$$=\frac{1}{3}u^{\frac{3}{2}}+C$$
$$=\underline{\frac{1}{3}\left(x^2+1\right)^{\frac{3}{2}}+C}$$

$u=x^2+1$

$du=2xdx$

$\frac{1}{2}du=xdx$

15) $\int \frac{\sin x}{1+2\cos x}dx$

$$\int -\frac{1}{2}\left(\frac{1}{u}\right)du$$
$$=-\frac{1}{2}\ln|u|+C$$
$$=\underline{-\frac{1}{2}\ln|1+2\cos x|+C}$$

$u=1+2\cos x$

$du=-2\sin x dx$

$-\frac{1}{2}du=\sin x dx$

16) $\int_0^3 \frac{x}{\sqrt{x+1}}dx.$

$$\int_1^4 \frac{u-1}{u^{\frac{1}{2}}}du$$
$$=\int_1^4 u^{\frac{1}{2}}-u^{-\frac{1}{2}}du$$
$$=\left[\frac{2}{3}u^{\frac{3}{2}}-2u^{\frac{1}{2}}\right]_1^4$$
$$=\frac{16}{3}-4-\frac{2}{3}+2$$
$$=\frac{14}{3}-2$$
$$=\frac{14}{3}-\frac{6}{3}=\underline{\frac{8}{3}}$$

$u=x+1$

$du=dx$

$x=u-1$

Limits

$x=0 \quad u=1$

$x=3 \quad u=4$

Worksheet: Integration by Substitution 6 **SCHOLAR: p94 Q54 - 65**

Advanced Higher Integration by Substitution 6

Unit 1 Outcome 1.3

1. Make use of a suitable substitution to find:

a) $\int x(x^2+1)^6 dx$

b) $\int x^2(x^3-2)^4 dx$

c) $\int x\sqrt{x^2+4}\,dx$

d) $\int \frac{x}{2x^2+1}dx$

e) $\int \frac{2x^3}{x^4+1}dx$

f) $\int \frac{x^2}{\sqrt{1+x^3}}dx$

g) $\int e^{2x}(e^{2x}+1)^3 dx$

h) $\int \frac{\sin x}{2\cos x+3}dx$

i) $\int x^2\sqrt{x^3+1}\,dx$

j) $\int x(x+1)^4 dx$

k) $\int x\sqrt{x-3}\,dx$

l) $\int \frac{x}{\sqrt{x+2}}dx$

m) $\int x^3(x^2+1)^6 dx$

n) $\int x^2\sqrt{x-1}\,dx$

o) $\int \frac{x^2}{\sqrt{x-2}}dx$

2. Make use of a suitable substitution to evaluate:

a) $\int_0^1 x(x^2+1)^4 dx$

b) $\int_0^2 x^2\sqrt{x^3+1}\,dx$

c) $\int_0^1 \frac{2x}{x^2+1}dx$

d) $\int_0^4 \frac{x}{\sqrt{x^2+9}}dx$

e) $\int_0^1 x(x+1)^4 dx$

f) $\int_0^5 x\sqrt{x+4}\,dx$

g) $\int_5^{10} \frac{x}{\sqrt{x-1}}dx$

h) $\int_0^{\frac{\pi}{4}} \sin 2x\sqrt{1-\cos 2x}\,dx$

i) $\int_0^3 \frac{x^2}{\sqrt{x+1}}dx$

3. Make use of the substitution $x = 3\sin\theta$ to show that $\int_0^{\frac{3}{2}} \frac{x+1}{\sqrt{9-x^2}}dx = 3-\frac{3\sqrt{3}}{2}+\frac{\pi}{6}$.

ANSWERS

1.

a) $\frac{1}{14}\left(x^2+1\right)^7+C$

b) $\frac{1}{15}\left(x^3-2\right)^5+C$

c) $\frac{1}{3}\left(x^2+4\right)^{\frac{3}{2}}+C$

d) $\frac{1}{4}\ln\left(2x^2+1\right)+C$

e) $\frac{1}{2}\ln\left(x^4+1\right)+C$

f) $\frac{2}{3}\sqrt{1+x^3}+C$

g) $\frac{1}{8}\left(e^{2x}+1\right)^4+C$

h) $-\frac{1}{2}\ln\left(2\cos x+3\right)+C$

i) $\frac{2}{9}\left(x^3+1\right)^{\frac{3}{2}}+C$

j) $\frac{1}{6}\left(x+1\right)^6-\frac{1}{5}\left(x+1\right)^5+C$

k) $\frac{2}{5}\left(x-3\right)^{\frac{5}{2}}+2(x-3)^{\frac{3}{2}}+C$

l) $\frac{2}{3}\left(x+2\right)^{\frac{3}{2}}-4\sqrt{x+2}+C$

m) $\frac{1}{16}\left(x^2+1\right)^8-\frac{1}{14}(x^2+1)^7+C$

n) $\frac{2}{7}\left(x-1\right)^{\frac{7}{2}}+\frac{4}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+C$

o) $\frac{2}{5}\left(x-2\right)^{\frac{5}{2}}+\frac{8}{3}(x-2)^{\frac{3}{2}}+8\sqrt{x-2}+C$

2.

a) $3\frac{1}{10}$

b) $5\frac{7}{9}$

c) $\ln 2 \cong 0.693$

d) 2

e) $4\frac{3}{10}$

f) $33\frac{11}{15}$

g) $14\frac{2}{3}$

h) $\frac{1}{3}$

i) $5\frac{1}{15}$

3. Proof.

Inverse Trig Functions

Since we know that $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$ it follows

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

NOTE: It only applies to SIN and TAN.

NOT COS !!!!!!

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C \qquad \int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Examples

1) $\int \frac{1}{\sqrt{9-x^2}}dx$

$\int \frac{1}{\sqrt{3^2-x^2}}dx$

$\sin^{-1}\left(\frac{x}{3}\right) + c$

2) $\int \frac{1}{16+x^2}dx$

$\int \frac{1}{4^2+x^2}dx$

$\frac{1}{2}\tan^{-1}\left(\frac{x}{4}\right) + c$

3) $\int \frac{1}{\sqrt{5-x^2}}dx$

$\int \frac{1}{\sqrt{\left(\sqrt{5}\right)^2-x^2}}dx$

$\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + c$

4) $\int \frac{4}{x^2+36}dx$

$\int 4\left(\frac{1}{6^2+x^2}\right)dx$

$4\left(\frac{1}{6}\right)\tan^{-1}\left(\frac{x}{6}\right) + c$

$\frac{2}{3}\tan^{-1}\left(\frac{x}{6}\right) + c$

Inverse Trig Functions

5) $\int \frac{1}{\sqrt{25-9x^2}}dx$

$\int \frac{1}{\sqrt{9\left(\frac{25}{9}-x^2\right)}}dx$

$\int \frac{1}{\sqrt{9\left(\left(\frac{5}{3}\right)^2-x^2\right)}}dx$

$\int \frac{1}{3\sqrt{\left(\left(\frac{5}{3}\right)^2-x^2\right)}}dx$

$\frac{1}{3}\sin^{-1}\left(\frac{x}{5/3}\right)+c$

$\underline{\underline{\frac{1}{3}\sin^{-1}\left(\frac{3x}{5}\right)+c}}$

6) $\int \frac{1}{9+4x^2}dx$

$\int \frac{1}{4\left(\frac{9}{4}+x^2\right)}dx$

$\int \frac{1}{4\left(\left(\frac{3}{2}\right)^2+x^2\right)}dx$

$\frac{1}{4}\left(\frac{2}{3}\right)\tan^{-1}\left(\frac{x}{3/2}\right)+c$

$\underline{\underline{\frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)+c}}$

$$\boxed{\int \frac{1}{\sqrt{a^2-(bx+c)^2}}dx = \frac{1}{b}\sin^{-1}\left(\frac{bx+c}{a}\right)+C}$$

$$\boxed{\int \frac{1}{a^2+(bx+c)^2}dx = \frac{1}{b}\left(\frac{1}{a}\right)\tan^{-1}\left(\frac{bx+c}{a}\right)+C}$$

7) $\int_{3/2}^{3} \frac{1}{\sqrt{9-x^2}}dx$

$\int_{3/2}^{3} \frac{1}{\sqrt{(3)^2-x^2}}dx$

$\left[\sin^{-1}\left(\frac{x}{3}\right)\right]_{3/2}^{3}$

$\sin^{-1}(1)-\sin^{-1}\left(\frac{3/2}{3}\right)$

$\sin^{-1}(1)-\sin^{-1}\left(\frac{1}{2}\right)$

$\frac{\pi}{2}-\frac{\pi}{6}$

$\underline{\underline{\frac{\pi}{3}}}$

8) $\int_{-2}^{2} \frac{1}{4+x^2}dx$

$\int_{-2}^{2} \frac{1}{(2)^2+x^2}dx$

$\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$

$\frac{1}{2}\left[\tan^{-1}(1)-\tan^{-1}(-1)\right]$

$\frac{1}{2}\left[\frac{\pi}{4}+\frac{\pi}{4}\right]$

$\underline{\underline{\frac{\pi}{4}}}$

Worksheet – Integration of Inverse Trig Functions 1 **Scholar p45 Q7 - 8**

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C \qquad \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

1. Make use of the standard integrals above to find each indefinite integral.

a) $\int \frac{1}{\sqrt{4-x^2}}\,dx$ b) $\int \frac{1}{\sqrt{16-x^2}}\,dx$ c) $\int \frac{1}{\sqrt{100-x^2}}\,dx$ d) $\int \frac{1}{9+x^2}\,dx$

e) $\int \frac{1}{25+x^2}\,dx$ f) $\int \frac{1}{4+x^2}\,dx$ g) $\int \frac{1}{\sqrt{25-x^2}}\,dx$ h) $\int \frac{2}{\sqrt{1-x^2}}\,dx$

i) $\int \frac{1}{36+x^2}\,dx$ j) $\int \frac{2}{16+x^2}\,dx$ k) $\int \frac{1}{\sqrt{49-x^2}}\,dx$ l) $\int \frac{4}{100+x^2}\,dx$

m) $\int \frac{1}{64+x^2}\,dx$ n) $\int \frac{6}{x^2+9}\,dx$ o) $\int \frac{1}{\sqrt{2-x^2}}\,dx$ p) $\int \frac{1}{3+x^2}\,dx$

q) $\int \frac{2}{2+x^2}\,dx$ r) $\int \frac{1}{\sqrt{1-4x^2}}\,dx$ s) $\int \frac{1}{1+9x^2}\,dx$ t) $\int \frac{1}{\sqrt{9-4x^2}}\,dx$

u) $\int \frac{1}{9+4x^2}\,dx$ v) $\int \frac{1}{\sqrt{16-25x^2}}\,dx$ w) $\int \frac{1}{16+9x^2}\,dx$ x) $\int \frac{1}{\sqrt{100-9x^2}}\,dx$

2. Evaluate each definite integral in terms of π:

a) $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$ b) $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$ c) $\int_0^{3} \frac{1}{\sqrt{36-x^2}}\,dx$ d) $\int_0^{1} \frac{1}{1+x^2}\,dx$

e) $\int_0^{4} \frac{1}{16+x^2}\,dx$ f) $\int_{\sqrt{3}}^{3} \frac{1}{9+x^2}\,dx$ g) $\int_0^{4} \frac{1}{\sqrt{64-x^2}}\,dx$ h) $\int_0^{3/2} \frac{1}{\sqrt{9-x^2}}\,dx$

i) $\int_{-2}^{2} \frac{1}{4+x^2}\,dx$ j) $\int_1^{\sqrt{3}} \frac{2}{1+x^2}\,dx$ k) $\int_{-1}^{\sqrt{2}} \frac{1}{\sqrt{4-x^2}}\,dx$ l) $\int_3^{6} \frac{4}{\sqrt{36-x^2}}\,dx$

m) $\int_0^{1} \frac{1}{\sqrt{2-x^2}}\,dx$ n) $\int_0^{3/4} \frac{1}{\sqrt{9-4x^2}}\,dx$

Answers

1.

a) $\sin^{-1}\left(\frac{x}{2}\right)+C$ b) $\sin^{-1}\left(\frac{x}{4}\right)+C$ c) $\sin^{-1}\left(\frac{x}{10}\right)+C$ d) $\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C$

e) $\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)+C$ f) $\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$ g) $\sin^{-1}\left(\frac{x}{5}\right)+C$ h) $2\sin^{-1}x+C$

i) $\frac{1}{6}\tan^{-1}\left(\frac{x}{6}\right)+C$ j) $\frac{1}{2}\tan^{-1}\left(\frac{x}{4}\right)+C$ k) $\sin^{-1}\left(\frac{x}{7}\right)+C$ l) $\frac{2}{5}\tan^{-1}\left(\frac{x}{10}\right)+C$

m) $\frac{1}{8}\tan^{-1}\left(\frac{x}{8}\right)+C$ n) $2\tan^{-1}\left(\frac{x}{3}\right)+C$ o) $\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$ p) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+C$

q) $\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$ r) $\frac{1}{2}\sin^{-1}2x+C$ s) $\frac{1}{3}\tan^{-1}(3x)+C$ t) $\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)+C$

u) $\frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)+C$ v) $\frac{1}{5}\sin^{-1}\left(\frac{5x}{4}\right)+C$ w) $\frac{1}{12}\tan^{-1}\left(\frac{3x}{4}\right)+C$ x) $\frac{1}{3}\sin^{-1}\left(\frac{3x}{10}\right)+C$

2.

a) $\frac{\pi}{6}$ b) $\frac{\pi}{6}$ c) $\frac{\pi}{6}$ d) $\frac{\pi}{4}$

e) $\frac{\pi}{16}$ f) $\frac{\pi}{36}$ g) $\frac{\pi}{6}$ h) $\frac{\pi}{6}$

i) $\frac{\pi}{4}$ j) $\frac{\pi}{6}$ k) $\frac{5\pi}{12}$ l) $\frac{4\pi}{3}$

m) $\frac{\pi}{4}$ n) $\frac{\pi}{12}$

Inverse Trig Functions

9) Use the substitution $u = x - 3$ to find the indefinite integral. $\int \frac{1}{x^2 - 6x + 13}dx$

$$u = x - 3$$
$$x = u + 3$$
$$x^2 = (u+3)^2$$
$$x^2 = u^2 + 6u + 9$$

$$\frac{du}{dx} = 1$$
$$du = dx$$

$$\int \frac{1}{u^2 + 6u + 9 - 6(u+3) + 13}du$$
$$\int \frac{1}{u^2 + 6u + 9 - 6u - 18 + 13}du$$
$$\int \frac{1}{u^2 + 4}du$$
$$\int \frac{1}{2^2 + u^2}du$$
$$\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$
$$\underline{\underline{\frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right) + C}}$$

10) Use the substitution $u = 2\sin x$ to evaluate the definite integral $\int_0^{\pi/6} \frac{\cos x}{1 + 4\sin^2 x}dx$

$$u = 2\sin x$$
$$u^2 = 4\sin^2 x$$
$$\frac{du}{dx} = 2\cos x$$
$$\frac{1}{2}du = \cos x dx$$

Limits

$$x = 0 \quad u = 2\sin 0 = 0$$
$$x = \frac{\pi}{6} \quad u = 2\sin\frac{\pi}{6} = 1$$

$$\int_0^1 \frac{1}{2}\left(\frac{1}{1+u^2}\right)du$$
$$\left[\frac{1}{2}\tan^{-1} u\right]_0^1$$
$$\frac{1}{2}\tan^{-1} 1 - \frac{1}{2}\tan^{-1} 0$$
$$\frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{2}(0)$$
$$\underline{\underline{\frac{\pi}{8}}}$$

Worksheet – Integration of Inverse Trig Functions 2 **Scholar p47 Q9 - 10**

1. Show that $\int_1^3 \frac{1}{3+x^2}\,dx = \frac{\pi}{6\sqrt{3}}$.

2. Use the substitution given in brackets to find each indefinite integral.

a) $\int \frac{1}{x^2+2x+5}\,dx \quad (u = x+1)$

b) $\int \frac{1}{x^2+4x+13}\,dx \quad (u = x+2)$

c) $\int \frac{1}{x^2-6x+10}\,dx \quad (u = x-3)$

d) $\int \frac{1}{x^2-4x+5}\,dx \quad (u = x-2)$

e) $\int \frac{1}{x^2+8x+20}\,dx \quad (u = x+4)$

f) $\int \frac{1}{x^2-2x+3}\,dx \quad (u = x-1)$

3. Make use of the substitution $u = x - 1$ to evaluate the definite integral $\int_1^4 \frac{1}{x^2-2x+10}\,dx$, expressing your answer in terms of π.

1. Proof

2.

a) $\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$

b) $\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+C$

c) $\tan^{-1}(x-3)+C$

d) $\tan^{-1}(x-2)+C$

e) $\frac{1}{2}\tan^{-1}\left(\frac{x+4}{2}\right)+C$

f) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+C$

3. $\frac{\pi}{12}$

Use of Partial Fractions and Algebraic Long Division

9) a) Express in partial fractions.

b) Hence find $\int \frac{x+7}{2x^2+3x-2}dx.$

$$\frac{x+7}{2x^2+3x-2} = \frac{A}{2x-1} + \frac{B}{x+2}$$

$$x+7 = A(x+2) + B(2x-1)$$

$x=-2 \qquad 5=-5B \qquad B=-1$

$x=\frac{1}{2} \qquad \frac{15}{2}=\frac{5}{2}A \qquad A=3$

$$\frac{x+7}{2x^2+3x-2} = \frac{3}{2x-1} - \frac{1}{x+2}$$

$$\int \frac{3}{2x-1} - \frac{1}{x+2}\,dx$$

$$\frac{3}{2}\ln|2x-1| - \ln|x+2| + c$$

10) a) Express $\frac{1}{x^2-x-6}$ in partial fractions.

b) Hence find the exact value of $\int_0^1 \frac{1}{x^2-x-6}dx$ giving your answer in the form $k \ln a$.

$$\frac{1}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$1 = A(+2) + B(x-3)$$

$x=3 \qquad 1=5A \qquad A=\frac{1}{5}$

$x=-2 \qquad 1=-5B \qquad B=-\frac{1}{5}$

$$\frac{1}{x^2-x-6} = \frac{1}{5(x-3)} - \frac{1}{5(x+2)}$$

$$\int_0^1 \frac{1}{5(x-3)} - \frac{1}{5(x+2)}\,dx$$

$$\frac{1}{5}\Big[\ln|x-3| - \ln|x+2|\Big]_0^1$$

$$\frac{1}{5}\Big[\ln|-2| - \ln|3| - \ln|-3| + \ln|2|\Big]$$

Magnitude therefore ignore negative sign

$$\frac{1}{5}\Big[\ln 2 + \ln 2 - (\ln 3 + \ln 3)\Big]$$

$$\frac{1}{5}\left[2\ln\frac{2}{3}\right]$$

$$\frac{2}{5}\ln\left(\frac{2}{3}\right)$$

Worksheet – Integration using Partial Fractions 1 **Scholar p51 Q14 - 15**

Advanced Higher Integration Using Partial Fractions 1 *Unit 1 Outcome 1.3*

1. Make use of partial fractions to find:

a) $\int \frac{3x}{(x-1)(x-2)}\,dx$

b) $\int \frac{x-3}{x(x-1)}\,dx$

c) $\int \frac{5x+1}{(x-1)(x+2)}\,dx$

d) $\int \frac{5-x}{(x-1)(x+3)}\,dx$

e) $\int \frac{3x+5}{(x+1)(x+3)}\,dx$

f) $\int \frac{x-8}{(x+1)(2x-1)}\,dx$

g) $\int \frac{2-3x}{x(1-x)}\,dx$

h) $\int \frac{1}{(2x+3)(x+2)}\,dx$

i) $\int \frac{2}{x^2-1}\,dx$

j) $\int \frac{3x-5}{x^2-3x+2}\,dx$

k) $\int \frac{x+6}{x^2+2x}\,dx$

l) $\int \frac{10}{2x^2+3x-2}\,dx$

m) $\int \frac{x-15}{2x^2-5x-12}\,dx$

n) $\int \frac{2(x+1)}{x^2+2x-3}\,dx$

o) $\int \frac{6x+5}{3x^2+5x-2}\,dx$

p) $\int \frac{x+7}{x^2+8x+15}\,dx$

q) $\int \frac{x}{x^2-2x-3}\,dx$

r) $\int \frac{x+1}{2x^2+x}\,dx$

2. Evaluate giving each answer correct to 3 decimal places:

a) $\int_4^6 \frac{4x-9}{(x-2)(x-3)}\,dx$

b) $\int_5^6 \frac{1}{(x-2)(x-4)}\,dx$

c) $\int_3^4 \frac{5}{(x+3)(x-2)}\,dx$

d) $\int_1^2 \frac{x+2}{x(x+4)}\,dx$

e) $\int_2^3 \frac{1}{(4-x)(x-1)}\,dx$

f) $\int_8^{12} \frac{6(x-5)}{(x-4)(x-7)}\,dx$

g) $\int_4^5 \frac{2x}{x^2-4x+3}\,dx$

h) $\int_5^6 \frac{x+24}{x^2-x-12}\,dx$

i) $\int_2^3 \frac{x-2}{2x^2-x-3}\,dx$

j) $\int_0^4 \frac{3x+7}{x^2+5x+6}\,dx$

k) $\int_2^6 \frac{x-11}{2x^2+x-3}\,dx$

l) $\int_2^4 \frac{3}{2x^2+5x+2}\,dx$

ANSWERS

1.

a) $-3\ln(x-1)+6\ln(x-2)+C$

b) $3\ln x-2\ln(x-1)+C$

c) $2\ln(x-1)+3\ln(x+2)+C$

d) $\ln(x-1)-2\ln(x+3)+C$

e) $\ln(x+1)+2\ln(x+3)+C$

f) $3\ln(x+1)-\frac{5}{2}\ln(2x-1)+C$

g) $2\ln x+\ln(1-x)+C$

h) $\ln(2x+3)-\ln(x+2)+C$

i) $\ln(x-1)-\ln(x+1)+C$

j) $2\ln(x-1)+\ln(x-2)+C$

k) $3\ln x-2\ln(x+2)+C$

l) $2\ln(2x-1)-2\ln(x+2)+C$

m) $\frac{3}{2}\ln(2x+3)-\ln(x-4)+C$

n) $\ln(x+3)+\ln(x-1)+C$

o) $\ln(3x-1)+\ln(x+2)+C$

p) $2\ln(x+3)-\ln(x+5)+C$

q) $\frac{3}{4}\ln(x-3)+\frac{1}{4}\ln(x+1)+C$

r) $\ln x-\frac{1}{2}\ln(2x+1)+C$

2.

a) 3.989

b) 0.203

c) 0.539

d) 0.438

e) 0.462

f) 7.824

g) 1.792

h) 2.419

i) 0.063

j) 2.793

k) -1.314

l) 0.182

Integrating an Improper Rational Function

Before integrating an improper rational function (a function where the power of the numerator is equal to or more than that of the denominator), long division should be used to express the function as a polynomial and proper rational function.

1) $\int \frac{x^2+4x+2}{x+1}dx$

$$\begin{array}{r|l} & \quad x+3 \\ \cline{2-2} x+1 & x^2+4x+2 \\ & \underline{x^2+x} \\ & \quad 3x+2 \\ & \quad \underline{3x+3} \\ & \qquad -1 \end{array}$$

$$\underline{\underline{x+3-\frac{1}{x+1}}}$$

$$\int \frac{x^2+4x+2}{x+1}dx$$

$$\int x+3-\frac{1}{x+1}dx$$

$$\underline{\underline{\frac{1}{2}x^2+3x-\ln|x+1|+c}}$$

Worksheet – Integration using Partial Fractions 2 **Scholar p54 Q16 - 18**

2) a) Express $\frac{x}{x^2-1}$ in partial fractions. b) Hence find $\int \frac{x^3}{x^2-1}dx.$

$$\frac{x}{x^2-1}=\frac{A}{x+1}+\frac{B}{x-1}$$

$$x=A(x-1)+B(x+1)$$

$$x=1 \qquad 1=2B \qquad B=\frac{1}{2}$$

$$x=-1 \quad -1=-2A \quad A=\frac{1}{2}$$

$$\underline{\underline{\frac{x}{x^2-1}=\frac{1}{2(x+1)}+\frac{1}{2(x-1)}}}$$

$$\begin{array}{r|l} & \quad x \\ \cline{2-2} x^2-1 & x^3 \\ & \underline{x^3-x} \\ & \qquad x \end{array}$$

$$x+\frac{x}{x^2-1}$$

$$\int x+\frac{x}{x^2-1}dx.$$

$$\int x+\frac{1}{2(x+1)}+\frac{1}{2(x-1)}dx$$

$$\underline{\underline{\frac{1}{2}x^2+\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|+c}}$$

Integrating an Improper Rational Function

a) Express $\dfrac{1}{x^3+x}$ as partial fractions.

b) Hence obtain a formula for $I(k)=\displaystyle\int_1^k \frac{1}{x^3+x}dx$, where $k>1$, expressing your formula in the form $\ln\left(\dfrac{a}{b}\right)$, where a and b are functions of k.

$$\frac{1}{x^3+x}$$

Worksheet – Integration using Partial Fractions 3

1. Using algebraic long division first, find:

a) $\int \frac{x}{x+2}\,dx$

b) $\int \frac{x}{x-1}\,dx$

c) $\int \frac{x}{x-4}\,dx$

d) $\int \frac{x}{x+6}\,dx$

e) $\int \frac{x+5}{x+2}\,dx$

f) $\int \frac{x+1}{x+3}\,dx$

g) $\int \frac{2x+3}{2x+1}\,dx$

h) $\int \frac{x^2+4x+2}{x+1}\,dx$

i) $\int \frac{x^2-x+3}{x-1}\,dx$

j) $\int \frac{x^2+6x+7}{x+4}\,dx$

k) $\int \frac{2x^2-3x+2}{x-2}\,dx$

l) $\int \frac{3x^2-2x+1}{3x+1}\,dx$

m) $\int \frac{x^3+5x^2+5x-1}{x+3}\,dx$

n) $\int \frac{2x^3-7x^2+7x-1}{2x-1}\,dx$

o) $\int \frac{2x^3+3x^2-2x-5}{x+1}\,dx$

p) $\int \frac{x^3-4x^2-3x+14}{x-4}\,dx$

q) $\int \frac{x-3}{x-1}\,dx$

r) $\int \frac{3x+2}{x-2}\,dx$

s) $\int \frac{2x}{x+3}\,dx$

t) $\int \frac{3x+1}{2x+1}\,dx$

u) $\int \frac{4x}{x-1}\,dx$

v) $\int \frac{2x^2+7x+1}{x+2}\,dx$

w) $\int \frac{x+2}{x-2}\,dx$

x) $\int \frac{x^2}{x+3}\,dx$

y) $\int \frac{x(2x+1)}{x+1}\,dx$

z) $\int \frac{x^2+4}{x+1}\,dx$

2. Evaluate giving each answer correct to 3 decimal places:

a) $\int_5^7 \frac{x}{x+5}\,dx$

b) $\int_3^5 \frac{x}{x-2}\,dx$

c) $\int_1^3 \frac{x+2}{x+1}\,dx$

d) $\int_3^8 \frac{x^2+7x+6}{x+2}\,dx$

e) $\int_0^4 \frac{x(x+3)}{x+2}\,dx$

f) $\int_1^3 \frac{x^2-2x-2}{x+1}\,dx$

g) $\int_1^3 \frac{2x^2+5x-1}{x+3}\,dx$

h) $\int_1^4 \frac{2x^2+5x+1}{2x+1}\,dx$

i) $\int_0^1 \frac{2x^2}{2x+1}\,dx$

j) $\int_4^{10} \frac{x^3+x^2-8x+8}{x-2}\,dx$

k) $\int_0^1 \frac{8x}{4x+3}\,dx$

l) $\int_0^1 \frac{x+1}{2x+1}\,dx$

ANSWERS

1.

a) $x-2\ln(x+2)+C$

b) $x+\ln(x-1)+C$

c) $x+4\ln(x-4)+C$

d) $x-6\ln(x+6)+C$

e) $x+3\ln(x+2)+C$

f) $x-2\ln(x+3)+C$

g) $x+\ln(2x+1)+C$

h) $\frac{1}{2}x^2+3x-\ln(x+1)+C$

i) $\frac{1}{2}x^2+3\ln(x-1)+C$

j) $\frac{1}{2}x^2+2x-\ln(x+4)+C$

k) $x^2+x+4\ln(x-2)+C$

l) $\frac{1}{2}x^2-x+\frac{2}{3}\ln(3x+1)+C$

m) $\frac{1}{3}x^3+x^2-x+2\ln(x+3)+C$

n) $\frac{1}{3}x^3-\frac{3}{2}x^2+2x+\frac{1}{2}\ln(2x-1)+C$

o) $\frac{2}{3}x^3+\frac{1}{2}x^2-3x-2\ln(x+1)+C$

p) $\frac{1}{3}x^3-3x-2\ln(x-4)+C$

q) $x-2\ln(x-1)+C$

r) $3x+8\ln(x-2)+C$

s) $2x-6\ln(x+3)+C$

t) $\frac{3}{2}x-\frac{1}{4}\ln(2x+1)+C$

u) $4x+4\ln(x-1)+C$

v) $x^2+3x-5\ln(x+2)+C$

w) $x+4\ln(x-2)+C$

x) $\frac{1}{2}x^2-3x+9\ln(x+3)+C$

y) $x^2-x+\ln(x+1)+C$

z) $\frac{1}{2}x^2-x+5\ln(x+1)+C$

2.

a) 1.088

b) 4.197

c) 2.693

d) 49.727

e) 9.803

f) -1.307

g) 6.811

h) 12.951

i) 0.275

j) 431.545

k) 0.729

l) 0.775

1. Using algebraic long division first and then partial fractions, find:

a) $\int \frac{x^2+2x+2}{x^2+x}\,dx$

b) $\int \frac{x^2}{x^2-4}\,dx$

c) $\int \frac{x^2+3x-13}{x^2+x-2}\,dx$

d) $\int \frac{x^2-4x-2}{x^2-2x}\,dx$

e) $\int \frac{2x^2+2x+3}{x^2-1}\,dx$

f) $\int \frac{x^2+2}{x^2+2x}\,dx$

g) $\int \frac{2x^3+4x^2-15x-8}{x^2+2x-8}\,dx$

h) $\int \frac{x^2+3}{x^2-1}\,dx$

i) $\int \frac{x^2+2x-6}{x^2+x-2}\,dx$

j) $\int \frac{x^3-x^2-5x+1}{x^2-2x-3}\,dx$

k) $\int \frac{x^3-5x^2+11x-12}{x^2-5x+6}\,dx$

l) $\int \frac{x^3+4x^2-x+2}{x^2+x}\,dx$

m) $\int \frac{x^2+2x-3}{x^2+3x+2}\,dx$

n) $\int \frac{x^3+2x^2-5x-6}{x^2-4x+3}\,dx$

o) $\int \frac{2x^3+x^2-3x+11}{2x^2+3x-2}\,dx$

1.

a) $x + 2\ln x - \ln(x+1) + C$

b) $x + \ln(x-2) - \ln(x+2) + C$

c) $x + 5\ln(x+2) - 3\ln(x-1) + C$

d) $x + \ln x - 3\ln(x-2) + C$

e) $2x + \frac{7}{2}\ln(x-1) - \frac{3}{2}\ln(x+1) + C$

f) $x + \ln x - 3\ln(x+2) + C$

g) $x^2 + 2\ln(x+4) - \ln(x-2) + C$

h) $x + 2\ln(x-1) - 2\ln(x+1) + C$

i) $x + 2\ln(x+2) - \ln(x-1) + C$

j) $\frac{1}{2}x^2 + x + \ln(x-3) - \ln(x+1) + C$

k) $\frac{1}{2}x^2 + 3\ln(x-3) + 2\ln(x-2) + C$

l) $\frac{1}{2}x^2 + 3x + 2\ln x - 6\ln(x+1) + C$

m) $x + 3\ln(x+2) - 4\ln(x+1) + C$

n) $\frac{1}{2}x^2 + 6x + 12\ln(x-3) + 4\ln(x-1) + C$

o) $\frac{1}{2}x^2 - x + 2\ln(2x-1) - \ln(x+2) + C$

INTEGRATION BY PARTS

When two functions $f(x)$ and $g(x)$ are multiplied, in order to DIFFERENTIATE, the PRODUCT RULE is used.

In order to integrate when two functions are multiplied, Integration by parts is used.

$$f(x) = u \qquad \frac{d}{dx}(u.v) = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$g(x) = v$$

$$u\frac{dv}{dx} = \frac{d}{dx}(u.v) - v\frac{du}{dx}$$

To integrate by parts

$$\int u\frac{dv}{dx} = u.v - \int v\frac{du}{dx}$$

$$\boxed{\int uv' = uv - \int u'v}$$

1) Use integration by parts to find $\int x\cos 2x\,dx$

2) Use integration by parts to find $\int x\sin 3x\,dx$

INTEGRATION BY PARTS

3) Use integration by parts to find $\int xe^{4x}dx$

4) Use integration by parts to find $\int (2x+1)e^{-2x}dx$

5) Use integration by parts to find $\int x^3 \ln x dx$

INTEGRATION BY PARTS

6) Use integration by parts to find $\int \frac{1}{x^4} \ln x \, dx$.

7) Use integration by parts to find $\int x(2x+1)^4 \, dx$

8) Use integration by parts to find $\int x\sqrt{2x-1} \, dx$.

INTEGRATION BY PARTS

9) Use integration by parts to find $\int \frac{x}{\sqrt{x+4}}\,dx.$

Integrating ln*x* and inverse trig functions.

Integration by parts can be used to integrate functions which can only be differentiated.

Set up the integral to include the term multiplied by 1.

10) Use integration by parts to find $\int \ln x\,dx$

Worksheet – Integration using Parts 1

1. Use the method of integration by parts to find each of these indefinite integrals.

a) $\int x\cos x\,dx$ b) $\int x\sin x\,dx$ c) $\int x\sin 2x\,dx$

d) $\int x\cos 3x\,dx$ e) $\int x\sin 4x\,dx$ f) $\int x\sin\frac{1}{2}x\,dx$

2. Use the method of integration by parts to find each of these indefinite integrals.

a) $\int xe^{x}dx$ b) $\int xe^{2x}dx$ c) $\int xe^{3x}dx$

d) $\int xe^{-x}dx$ e) $\int xe^{-4x}dx$ f) $\int (x+1)e^{4x}dx$

g) $\int (2x+3)e^{5x}dx$ h) $\int (1-3x)e^{-2x}dx$

3. Use the method of integration by parts to find each of these indefinite integrals.

a) $\int x\ln x\,dx$ b) $\int x^2\ln x\,dx$ c) $\int x^4\ln x\,dx$

d) $\int \frac{1}{x^2}\ln x\,dx$ e) $\int \frac{1}{x^3}\ln x\,dx$

4. Use the method of integration by parts to find each of these indefinite integrals.

a) $\int x(x+1)^3dx$ b) $\int x(x-1)^4dx$ c) $\int x(x+1)^5dx$

d) $\int x(2x+1)^3dx$ e) $\int x(3x+1)^4dx$ f) $\int (4x+1)(2x+3)^4dx$

g) $\int (2x+3)(2x-1)^5dx$ h) $\int (3x+1)(x+1)^8dx$

5. Use the method of integration by parts to find each of these indefinite integrals.

a) $\int x\sqrt{x+4}\,dx$ b) $\int x\sqrt{x-3}\,dx$ c) $\int \frac{x}{\sqrt{x+1}}dx$

d) $\int x\sqrt{2x+1}\,dx$ e) $\int \frac{x}{\sqrt{x-2}}dx$ f) $\int x\sqrt{3x+2}\,dx$

g) $\int x(2x+1)^{1/3}dx$ h) $\int \frac{2x+5}{\sqrt{3x-1}}dx$

1.

a) $x\sin x+\cos x+C$ b) $-x\cos x+\sin x+C$ c) $-\frac{1}{2}x\cos 2x+\frac{1}{4}\sin 2x+C$

d) $\frac{1}{3}x\sin 3x+\frac{1}{9}\cos 3x+C$ e) $-\frac{1}{4}x\cos 4x+\frac{1}{16}\sin 4x+C$ f) $-2x\cos\left(\frac{1}{2}x\right)+4\sin\left(\frac{1}{2}x\right)+C$

2.

a) xe^x-e^x+C b) $\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C$ c) $\frac{1}{3}xe^{3x}-\frac{1}{9}e^{3x}+C$

d) $-xe^{-x}-e^{-x}+C$ e) $-\frac{1}{4}xe^{-4x}-\frac{1}{16}e^{-4x}+C$ f) $\frac{1}{4}(x+1)e^{4x}-\frac{1}{16}e^{-4x}+C$

g) $\frac{1}{5}(2x+3)e^{5x}-\frac{2}{25}e^{5x}+C$ h) $-\frac{1}{2}(1-3x)e^{-2x}+\frac{3}{4}e^{-2x}+C$

3.

a) $\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$ b) $\frac{1}{3}x^3\ln x-\frac{1}{9}x^3+C$ c) $\frac{1}{5}x^5\ln x-\frac{1}{25}x^5+C$

d) $-\frac{1}{x}\ln x-\frac{1}{x}+C$ e) $-\frac{1}{2x^2}\ln x-\frac{1}{4x^2}+C$

4.

a) $\frac{1}{4}x(x+1)^4-\frac{1}{20}(x+1)^5+C$ b) $\frac{1}{5}x(x-1)^5-\frac{1}{30}(x-1)^6+C$ c) $\frac{1}{6}x(x+1)^6-\frac{1}{42}(x+1)^7+C$

d) $\frac{1}{8}x(2x+1)^4-\frac{1}{40}(2x+1)^5+C$ e) $\frac{1}{15}x(3x+1)^5-\frac{1}{270}(3x+1)^6+C$

f) $\frac{1}{10}(4x+1)(2x+3)^5-\frac{1}{30}(2x+3)^6+C$ g) $\frac{1}{12}(2x+3)(2x-1)^6-\frac{1}{84}(2x-1)^7+C$

h) $\frac{1}{9}(3x+1)(x+1)^9-\frac{1}{30}(x+1)^{10}+C$

5.

a) $\frac{2}{3}x(x+4)^{\frac{3}{2}}-\frac{4}{15}(x+4)^{\frac{5}{2}}+C$ b) $\frac{2}{3}x(x-3)^{\frac{3}{2}}-\frac{4}{15}(x-3)^{\frac{5}{2}}+C$

c) $2x\sqrt{x+1}-\frac{4}{3}(x+1)^{\frac{3}{2}}+C$ d) $\frac{1}{3}x(2x+1)^{\frac{3}{2}}-\frac{1}{15}(2x+1)^{\frac{5}{2}}+C$

e) $2x\sqrt{x-2}-\frac{4}{3}(x-2)^{\frac{3}{2}}+C$ f) $\frac{2}{9}x(3x+2)^{\frac{3}{2}}-\frac{4}{135}(3x+2)^{\frac{5}{2}}+C$

g) $\frac{3}{8}x(2x+1)^{\frac{4}{3}}-\frac{9}{112}(2x+1)^{\frac{7}{3}}+C$ h) $\frac{2}{3}(2x+5)\sqrt{3x-1}-\frac{8}{27}(3x-1)^{\frac{3}{2}}+C$

INTEGRATION BY PARTS

11) Use integration by parts to find $\int \sin^{-1} x dx$.

12) Use integration by parts to find $\int \tan^{-1} x dx$

13) Use integration by parts twice to find $\int x^2 \cos 3x dx$

INTEGRATION BY PARTS

14) Use integration by parts to evaluate the definitive integral $\int_0^{\pi/2} x \sin 2x.dx.$

15) Use integration by parts to evaluate the definitive integral $\int_0^1 xe^{-2x}dx,$ expressing your answer in terms of e.

Worksheet – Integration using Parts 2

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1. $\int \ln x\,dx$ can also be found using the method of integration by parts by noting that

$\int \ln x\,dx = \int (\ln x).1\,dx.$

Now let $u = \ln x$ and $\frac{dy}{dx} = 1$, and complete the integration.

2. a) Given $y = \ln(1+x^2)$, find $\frac{dy}{dx}$.

b) $\int \tan^{-1} x\,dx$ can be found using the method of integration by parts by noting that

$\int \tan^{-1} x\,dx = \int (\tan^{-1} x).1\,dx.$

Now let $u = \tan^{-1} x$ and $\frac{dy}{dx} = 1$, and complete the integration.

(you will need to make use of your answer to part a).

3.a) Given $y = xe^x$, find $\frac{dy}{dx}$ in its fully factorised form.

b) Hence use the method of integration by parts to find $\int \frac{xe^x}{(x+1)^2}\,dx.$

4. Use successive integration by parts to find each of these indefinite integrals.

a) $\int x^2 e^x\,dx$ b) $\int x^2 \sin x\,dx$ c) $\int x^2 \cos x\,dx$

d) $\int x^2 e^{-x}\,dx$ e) $\int x^2 e^{2x}\,dx$ f) $\int x^2 e^{3x}\,dx$

g) $\int x^2 \cos 2x\,dx$ h) $\int x^2 \sin 4x\,dx$ i) $\int x^2 (x+1)^5\,dx$

j) $\int x^2 \sqrt{x+1}\,dx$ k) $\int x^2 (2x+1)^4\,dx$ l) $\int x^2 \sqrt{2x+1}\,dx$

m) $\int x^2 \cos 3x\,dx$ n) $\int x^2 \sin 2x\,dx$

5. Use successive integration by parts to find $\int x^3 e^x\,dx.$

6. Use the method of integration by parts to evaluate each of these definite integrals.

a) $\int_0^{\frac{\pi}{2}} x \cos x\,dx$ b) $\int_0^{\pi} x \sin x\,dx$ c) $\int_0^{\frac{\pi}{4}} x \cos 2x\,dx$

d) $\int_0^{\pi} x \sin\left(\frac{1}{2}x\right)dx$ e) $\int_0^1 xe^x\,dx$ f) $\int_0^1 xe^{5x}\,dx$

g) $\int_0^1 xe^{-2x}\,dx$ h) $\int_0^1 x(x+1)^4\,dx$ i) $\int_0^3 x\sqrt{x+1}\,dx$

j) $\int_3^6 \frac{x}{\sqrt{x-2}}\,dx$

7. Use the method of integration by parts to evaluate each of these definite integrals correct to 3 dp.

a) $\int_1^2 x^3 \ln x\,dx$ b) $\int_1^4 \frac{1}{x^3} \ln x\,dx$

ANSWERS

1. $\int \ln x dx = x\ln x - x + C$

2. a) $\frac{dy}{dx} = \frac{2x}{1+x^2}$ b) $\int \tan^{-1} x dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$

3.a) $\frac{dy}{dx} = (x+1)e^x$ b) $\int \frac{xe^x}{(x+1)^2} dx = -\frac{xe^x}{x+1} + e^x + C$

4.

a) $x^2e^x - 2xe^x + 2e^x + C$

b) $-x^2\cos x + 2x\sin x + 2\cos x + C$

c) $x^2\sin x + 2x\cos x - 2\sin x + C$

d) $-x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

e) $\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$

f) $\frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$

g) $\frac{1}{2}x^2\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + C$

h) $-\frac{1}{4}x^2\cos 4x + \frac{1}{8}x\sin 4x - \frac{1}{32}\cos 4x + C$

i) $\frac{1}{6}x^2(x+1)^6 - \frac{1}{21}x(x+1)^7 + \frac{1}{168}(x+1)^8 + C$

j) $\frac{2}{3}x^2(x+1)^{\frac{3}{2}} - \frac{8}{15}x(x+1)^{\frac{5}{2}} + \frac{16}{105}(x+1)^{\frac{7}{2}} + C$

k) $\frac{1}{10}x^2(2x+1)^5 - \frac{1}{60}x(2x+1)^6 + \frac{1}{840}(2x+1)^7 + C$

l) $\frac{1}{3}x^2(2x+1)^{\frac{3}{2}} - \frac{2}{15}x(2x+1)^{\frac{5}{2}} + \frac{2}{105}2(x+1)^{\frac{7}{2}} + C$

m) $\frac{1}{3}x^2\sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + C$

n) $-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$

5. $\int x^3e^xdx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

6.

a) $\frac{\pi}{2} - 1$

b) π

c) $\frac{\pi}{8} - \frac{1}{4}$

d) 4

e) 1

f) $\frac{1}{25}(4e^5 + 1)$

g) $\frac{1}{4}(1 - 3e^{-2})$

h) $4\frac{3}{10}$

i) $7\frac{11}{15}$

j) $8\frac{2}{3}$

7. a) 1.835

b) 0.191

INTEGRATION BY PARTS

16) Use integration by parts twice to find $I = \int e^x \cos x dx$.

17) Define $I_n = \int_0^1 x^n e^{-x} dx$ for $n \geq 1$.

a) Use integration by parts to evaluate I_1.
b) Show that $I_n = nI_{n-1} - e^{-1}$ for $n \geq 2$.
c) Using your answers to part (a) and (b), evaluate I_3.

INTEGRATION BY PARTS

17) Define $I_n = \int_0^1 x^n e^{-x} dx$ for $n \geq 1$.

a) Use integration by parts to evaluate I_1.
b) Show that $I_n = nI_{n-1} - e^{-1}$ for $n \geq 2$.
c) Using your answers to part (a) and (b), evaluate I_3.

Continued