Calculus skills (1.2)

- **Differentiating functions using the chain rule.**
- **Differentiating functions using the product rule.**
- **Differentiating functions using the quotient rule.**
- **Differentiating inverse trigonometric functions.**
- **Finding the derivative of functions defined implicitly.**
- **Finding the derivative of functions defined parametrically.**

Recap

Differentiation is a method to calculate the rate at which a dependent output *y* changes with respect to the change in the independent input *x*.

This rate of change is called the **derivative** of *y* with respect to *x*.

The dependence of *y* upon *x* means that *y* is a function of *x*, denoted $y = f(x)$, where f is the function.

$f'(x)$ is the **DERIVED FUNCTION** – derivative of $f(x)$.

$f'(x)$ represents the **rate of change** of the function and the **gradient** of the tangent to the graph.

Function	***Derived Function***	***Examples***	
$f(x) = x^n$	$f'(x) = nx^{n-1}$	eg. $f(x) = x^5$	$f'(x) = 5x^4$
$f(x) = ax^n$	$f'(x) = anx^{n-1}$	eg. $f(x) = 4x^3$	$f'(x) = 12x^2$
$f(x) = (x+a)^n$	$f'(x) = n(x+a)^{n-1}$	eg. $f(x) = (x+3)^4$	$f'(x) = 4(x+3)^3$
$f(x) = g(x) + h(x)$	$f'(x) = g'(x) + h'(x)$	eg. $f(x) = 4x^3 - 15x^2 - 8x + 1$	$f'(x) = 12x^2 - 30x - 8$
$f(x) = \sin x$	$f'(x) = \cos x$		
$f(x) = \cos x$	$f'(x) = -\sin x$		

Chain Rule

$$f(x) = g\big(h(x)\big) \qquad f'(x) = g'\big(h(x)\big) \times h'(x)$$

Examples

1) $f(x) = (3x-2)^4$

$f'(x) = 4(3x-2)^3 \times 3$

$f'(x) = 12(3x-2)^3$

2) $f(x) = \cos^3(2x)$

$f(x) = (\cos(2x))^3$

$f'(x) = 3(\cos(2x))^2 \times (-2\sin(2x))$

$f'(x) = -6\sin 2x \cos^2 2x$

3) $f(x) = \dfrac{2}{(1-x)^3}$

$f(x) = 2(1-x)^{-3}$

$f'(x) = -3\big(2(1-x)^{-4}\big) \times -1$

$f'(x) = 6(1-x)^{-4}$

$f'(x) = \dfrac{6}{(1-x)^4}$

Differentiate the following:

4) $f(x) = (2x-7)^3$

$f(x) = (2x-7)^3$

$f'(x) = 3(2x-7)^2(2)$

$\underline{f'(x) = 6(2x-7)^2}$

5) $f(x) = (3x-1)^{-4}$

$f(x) = (3x-1)^{-4}$

$f'(x) = -4(3x-1)^{-5}(3)$

$f'(x) = -12(3x-1)^{-5}$

$\underline{f'(x) = -\dfrac{12}{(3x-1)^5}}$

6) $f(x) = \dfrac{4}{(x+3)^5}$

$f(x) = \dfrac{4}{(x+3)^5}$

$f(x) = 4(x+3)^{-5}$

$f'(x) = -20(x+3)^{-6}$

$\underline{f'(x) = -\dfrac{20}{(x+3)^6}}$

7) $f(x) = \dfrac{2}{\sqrt{(1+x)}}$

$f(x) = \dfrac{2}{\sqrt{(1+x)}}$

$f(x) = 2(1+x)^{-\frac{1}{2}}$

$f'(x) = -(1+x)^{-\frac{3}{2}}$

$\underline{f'(x) = -\dfrac{1}{(1+x)^{\frac{3}{2}}}}$

8) $f(x) = \cos^2(4x)$

$f(x) = \cos^2(4x)$

$f(x) = (\cos(4x))^2$

$f'(x) = 2\cos 4x(-4\sin 4x)$

$f'(x) = -4(2\sin 4x \cos 4x)$

$\sin 2A = 2\sin A \cos A$

$\underline{f'(x) = -4\sin 8x}$

9) $f(x) = \cos(x^3)$

$f(x) = \cos(x^3)$

$f'(x) = -\sin(x^3)(3x^2)$

$\underline{f'(x) = -3x^2\sin(x^3)}$

Notation (be familiar with the various ways that functions and their derivatives can be expressed.

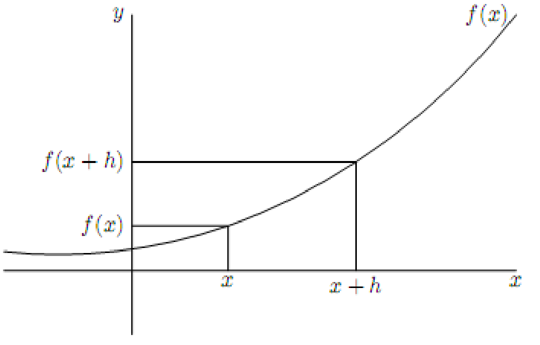
$$f(x) \to f'(x)$$
$$y \to \frac{dy}{dx}$$
$$u \to u'$$

Differentiation from first principles *(introduced in Higher)*

Finding the derivative is the same as finding the gradient of the tangent to the curve at a specific point.

Picking two points on the curve to find the gradient.

Points x and $x + h$. The smaller the gap (h), the more precise the approximation is for the gradient.



$$\boxed{\text{Gradient} = \frac{\text{difference in } y}{\text{difference in } x} = \frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}.}$$

This formula can be used to differentiate any function.

Examples

1) Find $f'(x)$ from first principles where $f(x) = 3x$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$f'(x) = \lim_{h\to 0}\frac{3(x+h)-3x}{h}$$
$$f'(x) = \lim_{h\to 0}\frac{3x+3h-3x}{h}$$
$$f'(x) = \lim_{h\to 0}\frac{3h}{h} = 3$$

2) Prove that for $f(x) = x^2$, $f'(x) = 2x$ using first principles.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$f'(x) = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$
$$f'(x) = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$
$$f'(x) = \lim_{h\to 0}\frac{h(2x+h)}{h} = \lim_{h\to 0}(2x+h)$$

As h tends to zero

$$f'(x) = 2x$$

Examples

Find the derivative of the following from first principles:

a) $f(x) = 3x^2$

$$f(x) = 3x^2$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3h(2x+h)}{h} = \lim_{h \to 0}(6x + 3h)$$

As h tends to zero

$$\underline{f'(x) = 6x}$$

b) $f(x) = \dfrac{2}{x}$

$$f(x) = \frac{2}{x}$$

$$f'(x) = \lim_{h \to 0} \frac{f\left(\frac{2}{x+h}\right) - \frac{2}{x}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\frac{2x - 2x - 2h}{x(x+h)}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\frac{-2h}{x(x+h)}}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{-2h}{hx(x+h)} = \lim_{h \to 0} \frac{-2}{x^2 + xh}$$

As h tends to zero

$$\underline{f'(x) = \frac{-2}{x^2}}$$

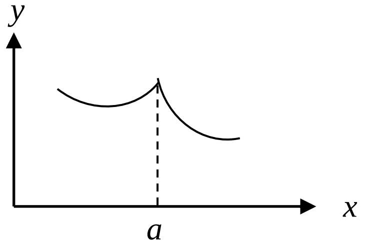
Scholar Page 44 Q4 – 6

Differentiability at a Point and Over an Interval

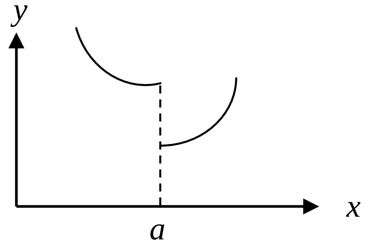
Scholar Page 45 – 48

A function can be differentiated at a point if there is a TANGENT at that point.

A function CANNOT be differentiated at a point if it is



NOT SMOOTH (corner)
No tangent

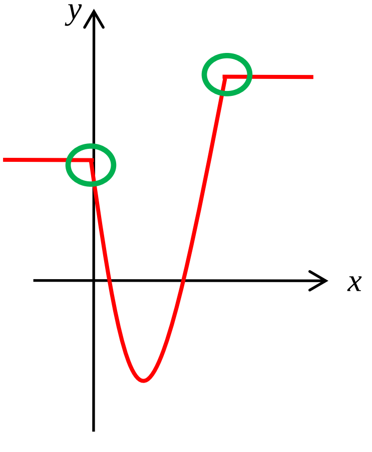


DISCONTINUOUS
No tangent

Over an interval - a function can only be differentiated over an interval [a, b] if tangents exist at both points *a* and *b*.

Examples

1) The function shown has two places where it is not differentiable.
Where and why?



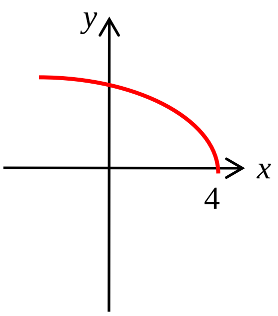
Corners – no tangent.

2. $f(x)=\sqrt{x+4}$ What is the largest possible domain for this function?

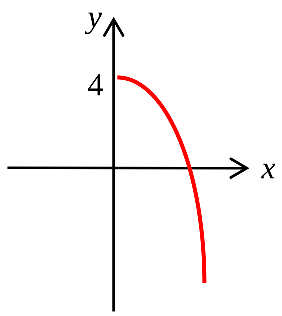
Cannot have $\sqrt{-}$ $\therefore x+4\geq 0$

$\underline{\underline{x\geq -4}}$

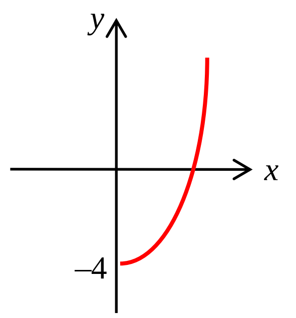
3. Identify which graphs represent $f(x)=\sqrt{x+4}$



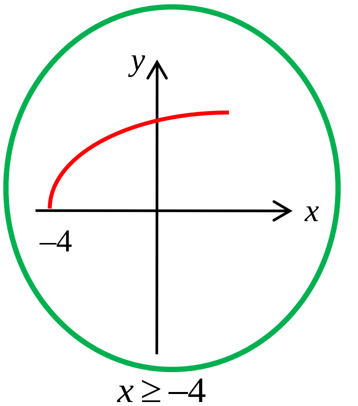
$x\leq 4$



$x\geq 0$



$x\geq 0$



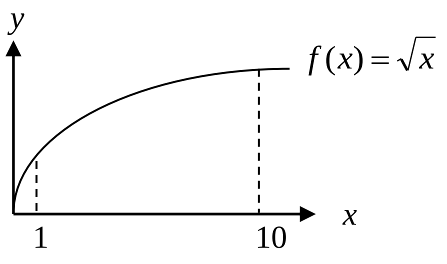
$x\geq -4$

Scholar Page 47 Q7 & 8

Differentiability Over an Interval

Closed Interval [a,b]

$f(x)$ can be differentiated over the interval $[a, b]$ only if $f'(x)$ (tangent) exists for a and b.



$f'(x)$ exists for (1, 10)

$f'(x)$ does not exist for $x = 0$ as the tangent tends to inifinity.

The Second Derivative and Rate of Change

For any displacement of an object (y), the speed over time and the acceleration can be found.

$$y \longrightarrow \frac{dy}{dx} \longrightarrow \frac{d^2y}{dx^2}$$

displacement velocity acceleration

$$\boxed{\frac{\text{change in displacement}}{\text{change in time}}} \quad \boxed{\frac{\text{change in velocity}}{\text{change in time}}}$$

Displacement (s)

Velocity (v)

$$v = \frac{ds}{dt}$$

Velocity is the change in distance over the change in time.

Acceleration (a)

$$a = \frac{dv}{dt}$$

Acceleration is the change in velocity over the change in time.

Example 1

The displacement of a particle s metres from a certain point at time t seconds is given by the formula

$$s(t) = 8 + 80t - 3t^3$$

a) Find the acceleration of the particle in terms of t.

b) Evaluate the acceleration after 5 seconds.

$$s(t) = 8 + 80t - 3t^3$$
$$v(t) = s'(t) = 80 - 9t^2$$
$$a(t) = s''(t) = -18t$$

$$t = 5$$
$$a(t) = -18t$$
$$a(t) = -18(5) = \underline{-90m/s^2}$$

<u>Example 2</u>

A particle is moving so that its distance, s metres, from the origin at time t seconds, is given by the formula

$$s(t) = \sin\left(6t - \frac{\pi}{6}\right)$$

where t belongs to the interval [0, 2π].

a) Calculate the velocity of the particle at t = 0.

b) Find an expression for the acceleration.

c) At what time t, belonging to the interval [0, 2π] is the acceleration at a maximum first. Give the answer as a multiple of π.

$$a) s(t) = \sin\left(6t - \frac{\pi}{6}\right)$$

$$v(t) = s'(t) = 6\cos\left(6t - \frac{\pi}{6}\right)$$

$$t = 0, v(0) = 6\cos\frac{\pi}{6}$$

$$v(0) = 5.196 m/s$$

$$b) a(t) = s''(t) = \underline{-36\sin\left(6t - \frac{\pi}{6}\right)}$$

$c)$ Maximum occurs when $\sin(A) = 1$ or -1.

$$-36\sin\left(6t - \frac{\pi}{6}\right) = 36$$

$$\left(6t - \frac{\pi}{6}\right) = -1 \qquad \sin\frac{3\pi}{2} = -1$$

$$6t - \frac{\pi}{6} = \frac{3\pi}{2}$$

$$6t = \frac{3\pi}{2} + \frac{\pi}{6}$$

$$6t = \frac{10\pi}{6}$$

$$t = \frac{10\pi}{36} = \underline{\frac{5\pi}{18}} \qquad \sin\left(6t - \frac{\pi}{6}\right) = \text{period} = 6$$

$$\text{Repeats every } \frac{2\pi}{6} = \frac{6\pi}{18} \qquad 360^o = 2\pi = \frac{36\pi}{18}$$

$$= \underline{\frac{5\pi}{18}, \frac{11\pi}{18}, \frac{17\pi}{18}, \frac{23\pi}{18}, \frac{29\pi}{18}, \frac{35\pi}{18}}.$$

Scholar Page 50 Q12 - 15

Higher Derivatives

We can continue differentiating the derivative terms.

Ex1. $f(x) = 4x^5$

$$\begin{aligned} f(x) &= 4x^5 \\ f'(x) &= 20x^4 \\ f''(x) &= 80x^3 \\ f'''(x) &= 240x^2 \\ f^{IV}(x) &= 480x \\ f^{V}(x) &= 480 \\ \underline{f^{VI}(x)} &\underline{= 0} \end{aligned}$$

Ex2. $y = 12x^3 + 8x^2 + 3x + 6$

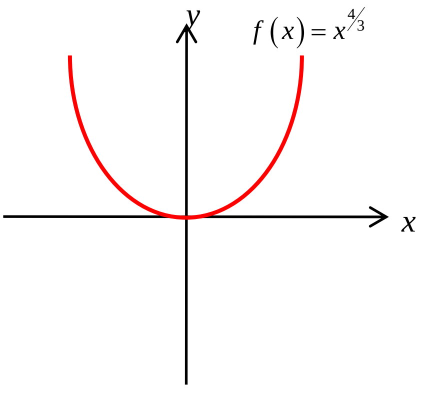
$$\begin{aligned} y &= 12x^3 + 8x^2 + 3x + 6 \\ \frac{dy}{dx} &= 36x^2 + 16x + 3 \\ \frac{d^2y}{dx^2} &= 72x + 16 \\ \frac{d^3y}{dx^3} &= 72 \\ \underline{\frac{d^4y}{dx^4}} &\underline{= 0} \end{aligned}$$

Scholar Page 51 Q16 & 17

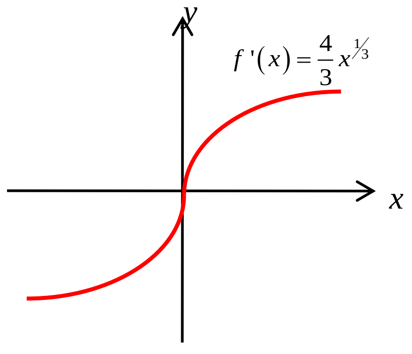
Discontinuities

We have already discussed points which cannot be differentiated eg. corners, discontinuities etc.

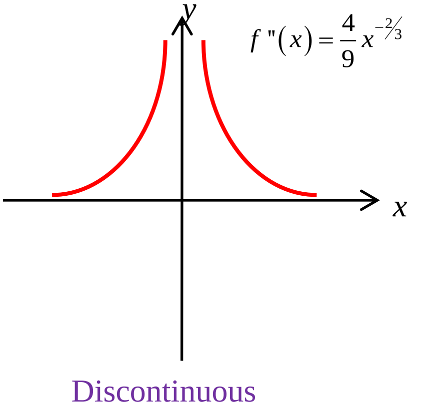
Derivatives of a function also can have discontinuities therefore be aware that there can be some x values for which it cannot be differentiated.



Continuous



Continuous
but not differentiable
at $x = 0$.



Discontinuous
at $x = 0$
(does not meet).

Scholar Page 52 Q18 & 19

TRIGONOMETRY

$$\frac{1}{\sin x} = \operatorname{cosec} x \qquad \frac{1}{\cos x} = \sec x \qquad \frac{1}{\tan x} = \cot x$$

$$\frac{\sin x}{\cos x} = \tan x$$

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2x = 2 \sin x \cos x$$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \end{aligned}$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

	0°	30° - $\frac{\pi}{6}$	45° - $\frac{\pi}{4}$	60° - $\frac{\pi}{3}$	90° - $\frac{\pi}{2}$
SIN	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
COS	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
TAN	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined

PRODUCT RULE

The product rule allows us to differentiate the product of 2 or more functions.

$$k(x) = f(x).g(x) \qquad k'(x) = f'(x).g(x) + f(x).g'(x)$$

$$y = u.v \qquad u' = \frac{du}{dx} \qquad v' = \frac{dv}{dx} \qquad \boxed{\frac{dy}{dx} = u'v + uv'}$$

1) $y = (3x+2)(5x-1)$

$u = 3x+2 \qquad v = 5x-1$

$u' = 3 \qquad v' = 5$

$$\frac{dy}{dx} = u'v + uv'$$
$$\frac{dy}{dx} = 3(5x-1) + 5(3x+2)$$
$$\frac{dy}{dx} = 15x - 3 + 15x + 10$$
$$\underline{\frac{dy}{dx} = 30x + 7}$$

2) $y = (x^2+1)(x^2+2x+3)$

$u = x^2+1 \qquad v = x^2+2x+3$

$u' = 2x \qquad v' = 2x+2$

$$\frac{dy}{dx} = u'v + uv'$$
$$\frac{dy}{dx} = 2x\left(x^2+2x+3\right) + \left(2x+2\right)\left(x^2+1\right)$$
$$\frac{dy}{dx} = 2x^3 + 4x^2 + 6x + 2x^3 + 2x + 2x^2 + 2$$
$$\underline{\frac{dy}{dx} = 4x^3 + 6x^2 + 8x + 2}$$

3) $y = x^6 \sin x$

$u = x^6 \qquad v = \sin x$

$u' = 6x^5 \qquad v' = \cos x$

$$\frac{dy}{dx} = u'v + uv'$$
$$\frac{dy}{dx} = 6x^5(\sin x) + x^6 \cos x$$
$$\underline{\frac{dy}{dx} = x^5\left(6\sin x + x\cos x\right)}$$

Worksheet – Product Rule 1 **Scholar Page 55 Q20 - 29**

1. Differentiate with respect to x, expressing each answer in its fully factorised form.

a) $y = x\sin x$ b) $y = x\cos x$ c) $y = x^2\sin x$ d) $y = x^4\cos x$

e) $y = x\sin 2x$ f) $y = x\cos 3x$ g) $y = x^2\sin 4x$ h) $y = x\cos(x^2)$

i) $y = x^2\sin(x^3)$ j) $y = x\cos 2x$ k) $y = \sin 2x\sin x$ l) $y = x^4\sin 3x$

m) $y = x^2\cos 2x$ n) $y = x\sin^2 x$ o) $y = x\cos^2 x$ p) $y = x\sin^3 x$

q) $y = x^2\sin^2 x$ r) $y = 2x^2\cos x$ s) $y = \sin 2x\cos 3x$ t) $y = x^2\sin 2x$

2. Given $f(x) = x\sin(\pi x)$, find the value of $f'\left(\frac{1}{2}\right)$.

3. Given $y = \sin x\cos x$, show that $\frac{dy}{dx} = \cos 2x$.

4.a) Differentiate $\sin x\cos 2x$ with respect to x.

b) Hence or otherwise, find $\frac{dy}{dx}$ given $y = x^3\sin x\cos 2x$.

5. Differentiate with respect to x, expressing each answer in its fully factorised form.

a) $y = x(x+1)^3$ b) $y = x(x-1)^4$ c) $y = x(x^2+1)^3$ d) $y = x^2(x+1)^4$

e) $y = x(x+2)^5$ f) $y = x(2x+1)^3$ g) $y = x(3x-1)^4$ h) $y = x^2(2x+3)^5$

i) $y = x^2(x^2+4)^3$ j) $y = x^3(x+2)^4$ k) $y = x^2(2x-1)^3$ l) $y = x^2(x^3+1)^4$

m) $y = x(2x+3)^3$ n) $y = (x+1)^2(x-1)^4$ o) $y = (x+1)^3(x+2)^4$ p) $y = (x+1)^4(x-1)^3$

q) $y = (x+1)^2(x+7)^5$ r) $y = (2x+1)^2(3x-1)^4$

6. Given $y = (x+3)^4(x-3)^5$, show that $\frac{dy}{dx} = 3(3x+1)(x+3)^3(x-3)^4$.

7.a) Given $y = x^2(x-3)^4$, show that $\frac{dy}{dx} = 6x(x-1)(x-3)^3$.

b) Find the coordinates and nature of each of the stationary points on the curve $y = x^2(x-3)^4$

8. Differentiate with respect to x, expressing each answer as a single algebraic fraction in its simplest form.

a) $y = x\sqrt{x+1}$ b) $y = x\sqrt{x+4}$ c) $y = x\sqrt{2x+3}$ d) $y = x\sqrt{3x+1}$

e) $y = x\sqrt{x^2+1}$ f) $y = x^2\sqrt{x-1}$ g) $y = x\sqrt{x^2+4}$ h) $y = x^3\sqrt{x+1}$

i) $y = x^2\sqrt{x^2+1}$ j) $y = \sqrt{x}(x+1)^3$ k) $y = \sqrt{x}(x-1)^2$ l) $y = \sqrt{x}(x+2)^4$

ANSWERS

1.

a) $x\cos x+\sin x$ b) $\cos x-x\sin x$ c) $x(x\cos x+2\sin x)$

d) $x^3(4\cos x-x\sin x)$ e) $2x\cos 2x+\sin 2x$ f) $\cos 3x-3x\sin 3x$

g) $2x(2x\cos 4x+\sin 4x)$ h) $\cos(x^2)-2x^2\sin(x^2)$ i) $x\{3x^3\cos(x^3)+2\sin(x^3)\}$

j) $\cos 2x-2x\sin 2x$ k) $\sin 2x\cos x+2\cos 2x\sin x$ l) $x^3(3x\cos 3x+4\sin 3x)$

m) $2x(\cos 2x-x\sin 2x)$ n) $\sin x(2x\cos x+\sin x)$ o) $\cos x(\cos x-2x\sin x)$

p) $\sin^2 x(3x\cos x+\sin x)$ q) $2x\sin x(x\cos x+\sin x)$ r) $2x(2\cos x-x\sin x)$

s) $2\cos 2x\cos 3x-3\sin 2x\sin 3x$ t) $2x(x\cos 2x+\sin 2x)$

2. $f'\left(\frac{1}{2}\right)=1$

3. Proof

4.a) $\cos x\cos 2x-2\sin x\sin 2x$ b) $x^2(x\cos x\cos 2x-2x\sin x\sin 2x+3\sin x\cos 2x)$

5.

a) $(4x+1)(x+1)^2$ b) $(5x-1)(x-1)^3$ c) $(7x^2+1)(x^2+1)^2$

d) $2x(3x+1)(x+1)^3$ e) $2(3x+1)(x+2)^4$ f) $(8x+1)(2x+1)^2$

g) $(15x-1)(3x-1)^3$ h) $2x(7x+3)(2x+3)^4$ i) $8x(x^2+1)(x^2+4)^2$

j) $x^2(7x+6)(x+2)^3$ k) $2x(5x-1)(2x-1)^2$ l) $2x(7x^3+1)(x^3+1)^3$

m) $(8x+3)(2x+3)^2$ n) $2(3x+1)(x+1)(x-1)^3$ o) $(7x+10)(x+1)^2(x+2)^3$

p) $(7x-1)(x+1)^3(x-1)^2$ q) $(7x+19)(x+1)(x+7)^4$ r) $4(9x+2)(2x+1)(3x-1)^3$

6. Proof

7.a) Proof b) min TP (0,0); max TP (1,16); min TP (3,0)

8.

a) $\dfrac{3x+2}{2\sqrt{x+1}}$ b) $\dfrac{3x+8}{2\sqrt{x+4}}$ c) $\dfrac{3(x+1)}{\sqrt{2x+3}}$

d) $\dfrac{9x+2}{2\sqrt{3x+1}}$ e) $\dfrac{2x^2+1}{\sqrt{x^2+1}}$ f) $\dfrac{x(5x-4)}{2\sqrt{x-1}}$

g) $\dfrac{2(x^2+2)}{\sqrt{x^2+4}}$ h) $\dfrac{x^2(7x+6)}{2\sqrt{x+1}}$ i) $\dfrac{x(3x^2+2)}{\sqrt{x^2+1}}$

j) $\dfrac{(7x+1)(x+1)^2}{2\sqrt{x}}$ k) $\dfrac{(5x-1)(x-1)}{2\sqrt{x}}$ l) $\dfrac{(9x+2)(x+2)^3}{2\sqrt{x}}$

QUOTIENT RULE

The quotient rule allows us to differentiate algebraic fractions.

$$k(x) = \frac{f(x)}{g(x)} \qquad k'(x) = \frac{f'(x).g(x) - f(x).g'(x)}{\left(g(x)\right)^2}$$

$$y = \frac{u}{v} \qquad u' = \frac{du}{dx} \qquad v' = \frac{dv}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{u'v - uv'}{v^2}}$$

1) $y = \dfrac{3x+2}{5x-1}$

$u = 3x+2 \qquad v = 5x-1$

$u' = 3 \qquad v' = 5$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

$$\frac{dy}{dx} = \frac{3(5x-1) - 5(3x+2)}{(5x-1)^2}$$

$$\frac{dy}{dx} = \frac{15x - 3 - 15x - 10}{(5x-1)^2}$$

$$\frac{dy}{dx} = \frac{-13}{(5x-1)^2}$$

2) $y = \dfrac{x^3+5}{4x^3-3}$

$u = x^3+5 \qquad v = 4x^3-3$

$u' = 3x^2 \qquad v' = 12x^2$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

$$\frac{dy}{dx} = \frac{3x^2(4x^3-3) - 12x^2(x^3+5)}{(4x^3-3)^2}$$

$$\frac{dy}{dx} = \frac{12x^5 - 9x^2 - 12x^5 - 60x^2}{(4x^3-3)^2}$$

$$\frac{dy}{dx} = \frac{-69x^2}{(4x^3-3)^2}$$

3) $y = \left(\dfrac{2x}{1-x}\right)^2$

$u = 4x^2 \qquad v = \left(1-x\right)^2$

$u' = 8x \qquad v' = -2\left(1-x\right)$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

$$\frac{dy}{dx} = \frac{8x\left(1-x\right)^2 + 8x^2\left(1-x\right)}{(1-x)^4}$$

$$\frac{dy}{dx} = \frac{8x(1-x)(1-x+x)}{(1-x)^4}$$

$$\frac{dy}{dx} = \frac{8x(1-x)}{(1-x)^4}$$

$$\frac{dy}{dx} = \frac{8x}{(1-x)^3}$$

Worksheet – Quotient Rule 1

Scholar Page 58 Q30 - 36

1. Differentiate with respect to x, expressing each answer in its fully factorised form.

a) $y=\dfrac{x}{x+4}$ b) $y=\dfrac{3x}{x+2}$ c) $y=\dfrac{x+1}{2x+1}$ d) $y=\dfrac{x}{x^2+1}$

e) $y=\dfrac{x^2}{x+3}$ f) $y=\dfrac{x^2-1}{x^2+1}$ g) $y=\dfrac{x^2}{4-x}$ h) $y=\dfrac{2x-1}{3x+2}$

i) $y=\dfrac{x^4}{x+1}$ j) $y=\dfrac{x^2}{x^3+1}$ k) $y=\dfrac{x^4}{x^2+1}$ l) $y=\dfrac{2x^2}{x-2}$

m) $y=\dfrac{\sin x}{x}$ n) $y=\dfrac{x}{\cos x}$ o) $y=\dfrac{x^2}{\sin x}$ p) $y=\dfrac{x^3}{\sin x}$

q) $y=\dfrac{\sin x}{x^2}$ r) $y=\dfrac{x^4}{\cos x}$ s) $y=\dfrac{\sin 2x}{x^2}$ t) $y=\dfrac{\sin x}{\cos x}$

u) $y=\dfrac{1+\sin x}{1+\cos x}$ v) $y=\dfrac{\sin x}{\sin x+\cos x}$ w) $y=\dfrac{x^3-1}{x^3+1}$ x) $y=\dfrac{2x^2+3x-6}{x-2}$

2. Given $f(x)=\dfrac{x+1}{x^2+2}$, find the value of $f'(0)$.

3. Differentiate with respect to x, expressing each answer in its simplest form.

a) $y=\dfrac{x}{(x+1)^2}$ b) $y=\dfrac{x}{(x+2)^3}$ c) $y=\dfrac{(x+1)^2}{(x+2)^3}$ d) $y=\dfrac{(2x+1)^2}{(3x+1)^2}$

4. Differentiate with respect to x, expressing each answer in its simplest form.

a) $y=\dfrac{x}{\sqrt{x+1}}$ b) $y=\dfrac{x}{\sqrt{x-2}}$ c) $y=\dfrac{x^2}{\sqrt{x+1}}$

d) $y=\dfrac{x}{\sqrt{x^2+1}}$ e) $y=\dfrac{x}{\sqrt{2x+1}}$ f) $y=\dfrac{x^2}{\sqrt{x^3+1}}$

ANSWERS

1.

a) $\dfrac{4}{(x+4)^2}$ b) $\dfrac{6}{(x+2)^2}$ c) $-\dfrac{1}{(2x+1)^2}$ d) $\dfrac{(1-x)(1+x)}{(x^2+1)^2}$

e) $\dfrac{x(x+6)}{(x+3)^2}$ f) $\dfrac{4x}{(x^2+1)^2}$ g) $\dfrac{x(8-x)}{(4-x)^2}$ h) $\dfrac{7}{(3x+2)^2}$

i) $\dfrac{x^3(3x+4)}{(x+1)^2}$ j) $\dfrac{x(2-x^3)}{(x^3+1)^2}$ k) $\dfrac{2x^3(x^2+2)}{(x^2+1)^2}$ l) $\dfrac{2x(x-4)}{(x-2)^2}$

m) $\dfrac{x\cos x-\sin x}{x^2}$ n) $\dfrac{\cos x+x\sin x}{\cos^2 x}$ o) $\dfrac{x(2\sin x-x\cos x)}{\sin^2 x}$ p) $\dfrac{x^2(3\sin x-x\cos x)}{\sin^2 x}$

q) $\dfrac{x\cos x-2\sin x}{x^3}$ r) $\dfrac{x^3(4\cos x+x\sin x)}{\cos^2 x}$ s) $\dfrac{2(x\cos 2x-\sin 2x)}{x^3}$ t) $\dfrac{1}{\cos^2 x}$

u) $\dfrac{1+\cos x+\sin x}{(1+\cos x)^2}$ v) $\dfrac{1}{(\sin x+\cos x)^2}$ w) $\dfrac{6x^2}{(x^3+1)^2}$ x) $\dfrac{2x(x-4)}{(x-2)^2}$

2. $f'(0)=\dfrac{1}{2}$

3.

a) $\dfrac{1-x}{(x+1)^3}$ b) $\dfrac{2(1-x)}{(x+2)^4}$ c) $\dfrac{(x+1)(1-x)}{(x+2)^4}$ d) $-\dfrac{2(2x+1)}{(3x+1)^3}$

4.

a) $\dfrac{x+2}{2(x+1)^{3/2}}$ b) $\dfrac{x-4}{2(x-2)^{3/2}}$ c) $\dfrac{x(3x+4)}{2(x+1)^{3/2}}$

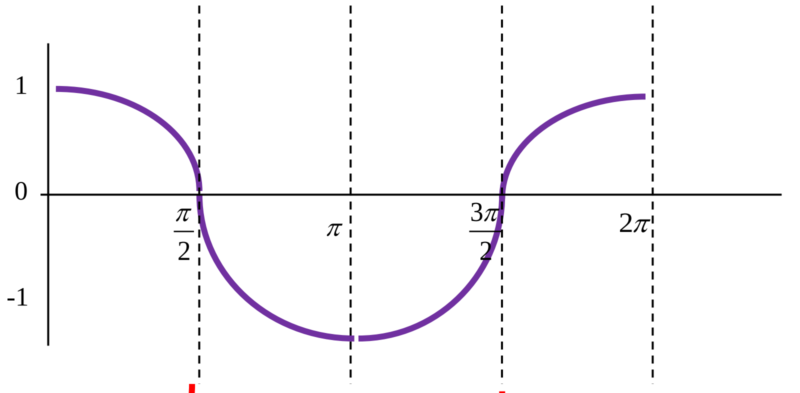
d) $\dfrac{1}{(x^2+1)^{3/2}}$ e) $\dfrac{x+1}{(2x+1)^{3/2}}$ f) $\dfrac{x(x^3+4)}{2(x^3+1)^{3/2}}$

NEW TRIGONOMETRY

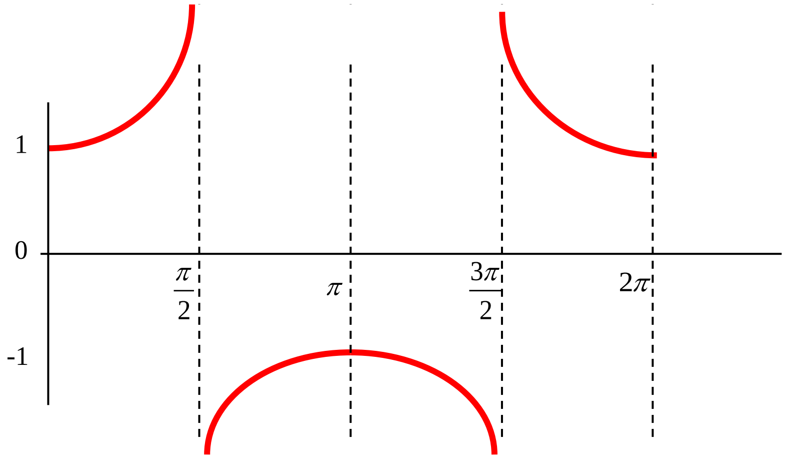
1) **SECANT (Sec)**

$$\text{Sec } x = \frac{1}{\cos x}$$

Cosine



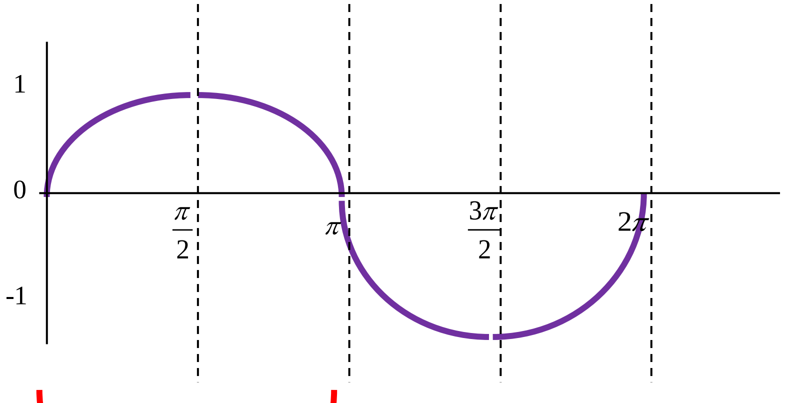
Secant



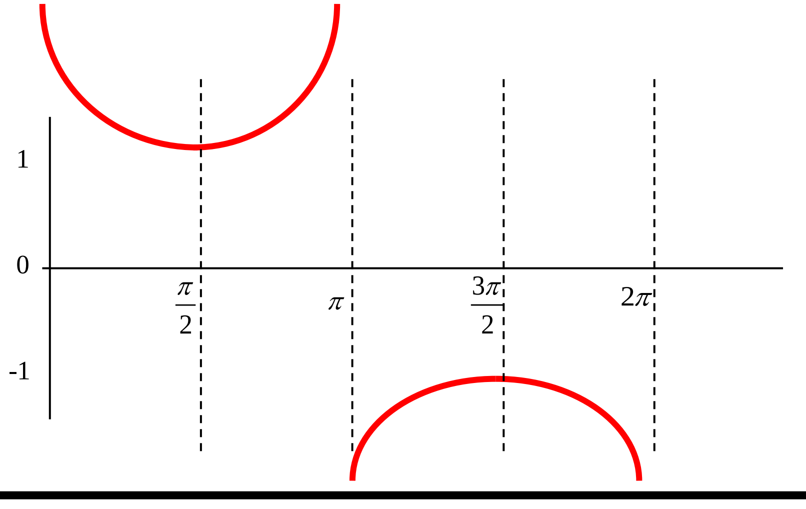
2) **COSECANT (Cosec)**

$$\text{Cosec } x = \frac{1}{\sin x}$$

Sin



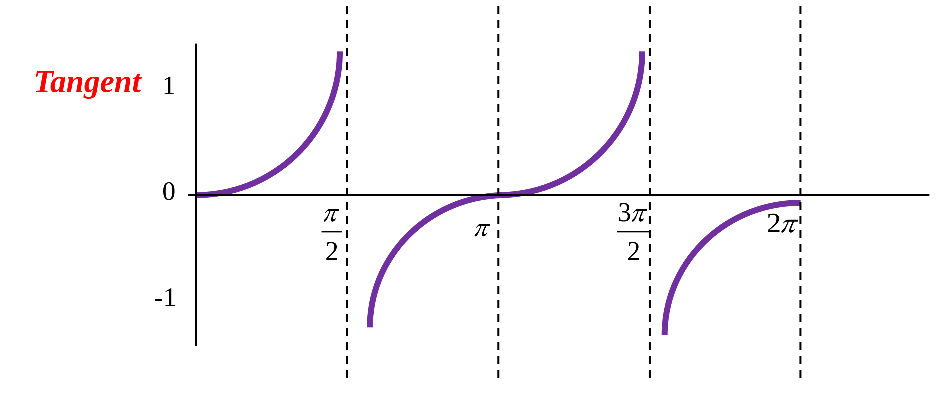
Cosec

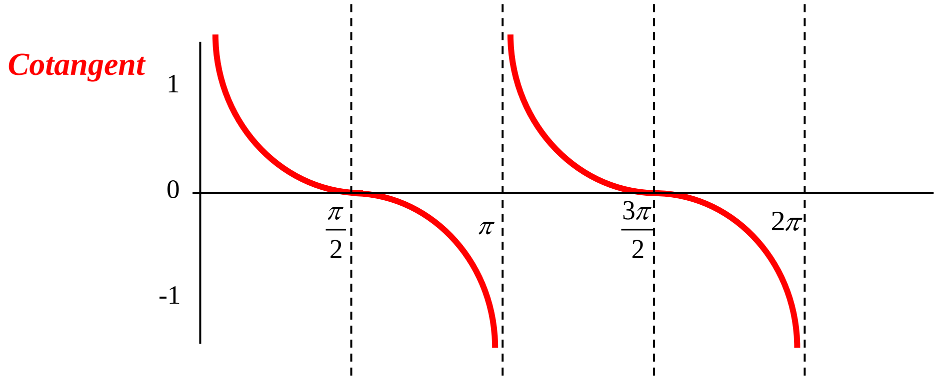


NEW TRIGONOMETRY

1) **COTANGENT (Cot)**

$$\boxed{\text{Cot } x = \frac{1}{\tan x}}$$





Examples

1) $\operatorname{cosec} \frac{\pi}{3}$

$$= \frac{1}{\sin \pi/3} = \frac{1}{\sqrt{3}/2} = \underline{\frac{2}{\sqrt{3}}}$$

2) $\sec \frac{\pi}{4}$

$$= \frac{1}{\cos \pi/4} = \frac{1}{1/\sqrt{2}} = \underline{\sqrt{2}}$$

3) $\cot^2 \frac{\pi}{6}$

$$= \frac{1}{\tan^2 \pi/6} = \frac{1}{\left(1/\sqrt{3}\right)^2} = \frac{1}{1/3} = \underline{3}$$

Derivative of Tan x

$y = \tan x \qquad y = \frac{\sin x}{\cos x}$

$u = \sin x \qquad u' = \cos x$

$v = \cos x \qquad v' = -\sin x$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

$$\frac{dy}{dx} = \frac{\cos x.\cos x - \sin x(-\sin x)}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$\cos^2 x + \sin^2 x = 1$

$$\frac{dy}{dx} = \frac{1}{\cos^2 x} = \sec^2 x$$

$\frac{1}{\cos x} = \sec x$

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

Derivative of Cosec x

$y = \text{cosec} x \qquad y = \frac{1}{\sin x}$

$u = 1 \qquad u' = \sin x$

$v = 0 \qquad v' = \cos x$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

$$\frac{dy}{dx} = \frac{0(\sin x) - \cos x}{(\sin x)(\sin x)} = \frac{-\cos x}{(\sin x)(\sin x)}$$

$$\frac{dy}{dx} = \left(\frac{-1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)$$

$$\frac{dy}{dx} = -\text{cosec} x.\cot x$$

$$\boxed{\frac{d}{dx}(\cos \text{ec} x) = -\text{cosec} x.\cot x}$$

Derivative of Sec x

$y = \sec x \qquad y = \frac{1}{\cos x}$

$u = 1 \qquad u' = \cos x$

$v = 0 \qquad v' = -\sin x$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

$$\frac{dy}{dx} = \frac{0(\cos x) - (-\sin x)}{(\cos x)(\cos x)} = \frac{\sin x}{(\cos x)(\cos x)}$$

$$\frac{dy}{dx} = \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right)$$

$$\frac{dy}{dx} = \sec x.\tan x$$

$$\boxed{\frac{d}{dx}(\sec x) = \sec x.\tan x}$$

Derivative of Cot x

$y = \cot x \qquad y = \frac{1}{\tan x}$

$u = \cos x \qquad u' = \sin x$

$v = -\sin x \qquad v' = \cos x$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

$$\frac{dy}{dx} = \frac{-\sin x(\sin x) - \cos x(\cos x)}{(\sin x)(\sin x)} = \frac{-(\sin^2 x + \cos^2 x)}{(\sin x)^2}$$

$$\frac{dy}{dx} = \frac{-1}{(\sin x)^2} = -\left(\frac{1}{\sin x}\right)^2$$

$$\frac{dy}{dx} = -\text{cosec}^2 x$$

$$\boxed{\frac{d}{dx}(\cot x) = -\text{cosec}^2 x}$$

Worksheet – Differentiation of Trig Functions

1. Use the chain rule to differentiate each function with respect to x:

a) $y=\sin 2x$ b) $y=\cos 4x$ c) $y=\tan 2x$ d) $y=\sec 3x$

e) $y=\operatorname{cosec}5x$ f) $y=\cot 2x$ g) $y=\sin(x^2)$ h) $y=\cos(1-2x)$

i) $y=\tan 4x$ j) $y=\sec(2x+3)$ k) $y=\operatorname{cosec}ax$ l) $y=\cot 3x$

m) $y=\sin^3 x$ n) $y=\cos^5 x$ o) $y=\tan^2 x$ p) $y=\sec^4 x$

q) $y=\operatorname{cosec}^2 x$ r) $y=\cot^3 x$ s) $y=\sin^2 3x$ t) $y=\cos^3 4x$

u) $y=\tan^3 2x$ v) $y=\sec^2 2x$ w) $y=\operatorname{cosec}^2 3x$ x) $y=\cot^2 5x$

y) $y=\tan^2 4x$ z) $y=\sec^6 3x$

2. Use the product rule to differentiate each function with respect to x and simplify where possible.

a) $y=x\tan x$ b) $y=x^2\sec x$ c) $y=x^2\cot x$

d) $y=\sec x\tan x$ e) $y=x^2\tan 2x$ f) $y=x^2\operatorname{cosec}2x$

g) $y=x^2\sec 4x$ h) $y=x\tan^2 x$ i) $y=x^2\sec^2 x$

3. Use the quotient rule to differentiate each function with respect to x and simplify where possible.

a) $y=\dfrac{x}{\tan x}$ b) $y=\dfrac{x}{\cot x}$ c) $y=\dfrac{\sec x}{x}$

d) $y=\dfrac{x^2}{\sec x}$ e) $y=\dfrac{x}{\operatorname{cosec}x}$ f) $y=\dfrac{\tan 2x}{x^2}$

4. Given $f(x)=\sin x\sec x$, show that $f'\left(\dfrac{\pi}{3}\right)=4.$

[You can either use the product rule to differentiate *f(x)* in the form above or you can simplify *f(x)* before differentiating.]

ANSWERS

1.

a) $2\cos 2x$ b) $-4\sin 4x$ c) $2\sec^2 2x$ d) $3\sec 3x \tan 3x$

e) $-5\operatorname{cosec} 5x \cot 5x$ f) $-2\operatorname{cosec}^2 2x$ g) $2x\cos(x^2)$ h) $2\sin(1-2x)$

i) $4\sec^2 4x$ j) $2\sec(2x+3)\tan(2x+3)$ k) $-a\operatorname{cosec} ax \cot ax$ l) $-3\operatorname{cosec}^2 3x$

m) $3\sin^2 x\cos x$ n) $-5\cos^4 x\sin x$ o) $2\sec^2 x\tan x$ p) $4\sec^4 x\tan x$

q) $-2\operatorname{cosec}^2 x\cot x$ r) $-3\cot^2 x\operatorname{cosec}^2 x$ s) $6\sin 3x\cos 3x$ t) $-12\cos^2 4x\sin 4x$

u) $6\tan^2 2x\sec^2 2x$ v) $4\sec^2 2x\tan 2x$ w) $-6\operatorname{cosec}^2 3x\cot 3x$ x) $-10\operatorname{cosec}^2 5x\cot 5x$

y) $8\sec^2 4x\tan 4x$ z) $18\sec^6 3x\tan 3x$

2.

a) $x\sec^2 + \tan x$ b) $x\sec x(x\tan x + 2)$ c) $x(2\cot x - x\operatorname{cosec}^2 x)$

d) $\sec x(\sec^2 x + \tan^2 x)$ e) $2x(x\sec^2 2x + \tan 2x)$ f) $x^2\operatorname{cosec} 2x(3 - 2x\cot 2x)$

g) $2x\sec 4x(2x\tan 4x + 1)$ h) $\tan x(2x\sec^2 x + \tan x)$ i) $2x\sec^2 x(x\tan x + 1)$

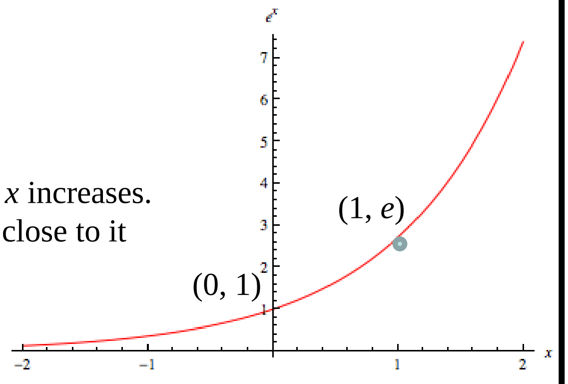
3.

a) $\dfrac{\tan x - x\sec^2 x}{\tan^2 x}$ b) $\dfrac{\cot x + x\operatorname{cosec}^2 x}{\cot^2 x}$ c) $\dfrac{\sec x(x\tan x - 1)}{x^2}$

d) $\dfrac{x(2 - x\tan x)}{\sec x}$ e) $\dfrac{1 + x\cot x}{\operatorname{cosec} x}$ f) $\dfrac{2(x\sec^2 2x - \tan 2x)}{x^3}$

Derivative of exp (x) or e^x

In mathematics, the **exponential function** is the function e^x, where e is the number (approximately 2.718281828).



The graph of $y = e^x$ is upward-sloping, and increases faster as x increases. The graph always lies above the x-axis but can get arbitrarily close to it for negative x; thus, the x-axis is a horizontal asymptote.

The slope of the graph at each point is equal to its y coordinate at that point.

The inverse function is the natural logarithm $\ln(x)$;

$$\boxed{f(x) = e^x \qquad f'(x) = e^x}$$

Examples

Find the derivative of the following:

a) $y = e^{3x}$

$$y = e^{3x}$$
$$\frac{dy}{dx} = 3e^{3x}$$

b) $y = xe^x$

$u = x$
$u' = 1$
$v = e^x$
$v' = e^x$

$$y = xe^x$$
$$\frac{dy}{dx} = u'v + uv'$$
$$\frac{dy}{dx} = e^x + xe^x$$
$$\frac{dy}{dx} = e^x(1+x)$$

c) $y = 3e^{x^2} + 4e^{-2x}$

$$y = e^{3x}$$
$$\frac{dy}{dx} = 3e^{x^2} + 4e^{-2x}$$
$$\frac{dy}{dx} = 3(2x)e^{2} - 2(4)e^{-2x}$$
$$\frac{dy}{dx} = 6xe^{x^2} - 8e^{-2x}$$

d) $y = e^{2x}\cos 2x$

$u = e^{2x}$
$u' = 2e^{2x}$
$v = \cos 2x$
$v' = -2\sin 2x$

$$y = xe^x$$
$$\frac{dy}{dx} = u'v + uv'$$
$$\frac{dy}{dx} = 2e^{2x}\cos 2x - 2e^{2x}\sin 2x$$
$$\frac{dy}{dx} = 2e^{2x}(\cos 2x - \sin 2x)$$

e) $y = \dfrac{x^2}{e^{2x}}$

$u = x^2$
$u' = 2x$
$v = e^{2x}$
$v' = 2e^{2x}$

$$y = xe^x$$
$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$
$$\frac{dy}{dx} = \frac{2xe^{2x} - 2x^2e^{2x}}{\left(e^{2x}\right)^2}$$
$$\frac{dy}{dx} = \frac{2xe^{2x}(1-x)}{\left(e^{2x}\right)^2}$$
$$\frac{dy}{dx} = \frac{2x(1-x)}{e^{2x}}$$

Scholar Page 61 Q43 - 51

Worksheet – Differentiation of Exp Functions

Advanced Higher Differentiation – Exp. Functions *Unit 1 Outcome 1.2*

1. Use the chain rule to differentiate each function with respect to x.

a) $y = e^{4x}$ b) $y = e^{x^2}$ c) $y = e^{\sin x}$ d) $y = 2e^{3x}$

e) $y = e^{-2x}$ f) $y = e^{x^3}$ g) $y = 8e^{\frac{1}{2}x}$ h) $y = e^{\cos x}$

i) $y = 4e^{-3x}$ j) $y = 2e^{-x}$ k) $y = e^{-3x^2}$ l) $y = e^{\sin 2x}$

m) $y = e^{2x^2+1}$ n) $y = 4e^{2x} + 3e^{-4x}$ o) $y = 3e^{x^2} - 5e^{-2x}$ p) $y = e^{\sqrt{x}}$

2. Use the product rule to differentiate each function with respect to x and simplify.

a) $y = xe^x$ b) $y = e^x \sin x$ c) $y = x^3 e^x$ d) $y = xe^{2x}$

e) $y = e^{-2x} \sin x$ f) $y = x^2 e^{4x}$ g) $y = 2x^3 e^{2x}$ h) $y = e^x \cos 2x$

i) $y = xe^{x^2}$ j) $y = x^2 e^{-2x}$ k) $y = e^{2x} \sin 4x$ l) $y = x^3 e^{3x}$

m) $y = (2x-1)e^{2x}$

3. Use the quotient rule to differentiate each function with respect to x and simplify.

a) $y = \dfrac{e^x}{x}$ b) $y = \dfrac{e^{2x}}{x}$ c) $y = \dfrac{x}{e^x}$ d) $y = \dfrac{e^x}{x+1}$

e) $y = \dfrac{e^x}{x^2}$ f) $y = \dfrac{\sin x}{e^x}$ g) $y = \dfrac{e^x}{x^3}$ h) $y = \dfrac{e^x}{\sin x}$

i) $y = \dfrac{x^2}{e^x}$ j) $y = \dfrac{e^{2x}}{\cos x}$ k) $y = \dfrac{x^2}{e^{3x}}$ l) $y = \dfrac{e^{2x}-1}{e^{2x}+1}$

m) $y = \dfrac{x(x+2)}{e^x}$

ANSWERS

1.

a) $4e^{4x}$ b) $2xe^{x^2}$ c) $\cos xe^{\sin x}$ d) $6e^{3x}$

e) $-2e^{-2x}$ f) $3x^2e^{x^3}$ g) $4e^{\frac{1}{2}x}$ h) $-\sin xe^{\cos x}$

i) $-12e^{-3x}$ j) $-2e^{-x}$ k) $-6xe^{-3x^2}$ l) $2\cos 2xe^{\sin 2x}$

m) $4xe^{2x^2+1}$ n) $8e^{2x}-12e^{-4x}$ o) $6xe^{x^2}+10e^{-2x}$ p) $\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

2.

a) $(x+1)e^x$ b) $e^x(\cos x+\sin x)$ c) $x^2(x+3)e^x$ d) $(2x+1)e^{2x}$

e) $e^{-2x}(\cos x-2\sin x)$ f) $2x(2x+1)e^{4x}$ g) $2x^2(2x+3)e^{2x}$ h) $e^x(\cos 2x-2\sin 2x)$

i) $(2x^2+1)e^{x^2}$ j) $2x(1-x)e^{-2x}$ k) $2e^{2x}(2\cos 4x+\sin 4x)$ l) $3x^2(x+1)e^{3x}$

m) $4xe^{2x}$

3.

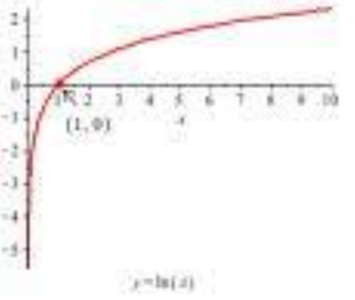
a) $\dfrac{(x-1)e^x}{x^2}$ b) $\dfrac{(2x-1)e^{2x}}{x^2}$ c) $\dfrac{1-x}{e^x}$ d) $\dfrac{xe^x}{(x+1)^2}$

e) $\dfrac{2(x-1)e^{2x}}{x^3}$ f) $\dfrac{\cos x-\sin x}{e^x}$ g) $\dfrac{(x-3)e^x}{x^4}$ h) $\dfrac{e^x(\sin x-\cos x)}{\sin^2 x}$

i) $\dfrac{x(2-x)}{e^x}$ j) $\dfrac{e^{2x}(2\cos x+\sin x)}{\cos^2 x}$ k) $\dfrac{x(2-3x)}{e^{3x}}$ l) $\dfrac{4e^{2x}}{(e^{2x}+1)^2}$

m) $\dfrac{2-x^2}{e^x}$

Derivative of ln(x)



The natural logarithm of a number x (written as *ln(x)*) is the power to which e would have to be raised to equal x.

The natural logarithm function, is the inverse function of the exponential function

$$\boxed{f(x) = \ln x \qquad f'(x) = \frac{1}{x}}$$

Rules of Logs:

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a x^n = n\log_a$$

Examples

Find the derivative of the following:

a) $y = \ln(3x\text{-}1)$

$$y' = \frac{1}{3x-1}(3)$$

$$y' = \frac{3}{3x-1}$$

b) $y = \ln\left(4x^2+1\right)$

$$y' = \frac{1}{4x^2+1}(8x)$$

$$y' = \frac{8x}{4x^2+1}$$

c) $f(x) = \ln\left(e^{2x}+1\right)$

$$y' = \frac{1}{e^{2x}+1}(2e^{2x})$$

$$y' = \frac{2e^{2x}}{e^{2x}+1}$$

d) $y = \ln(\cos 4x)$

$$y' = \frac{1}{\cos 4x}(-4\sin 4x)$$

$$y' = -\frac{4\sin 4x}{\cos 4x}$$

$$y' = -4\tan 4x$$

Derivative of ln(*x*)

e) $y = x^3 \ln x$

$u = x^3$

$u' = 3x^2$

$v = \ln x$

$v' = \frac{1}{x}$

$y' = u'v + uv'$

$y' = 3x^2 \ln x + \frac{x^3}{x}$

$y' = 3x^2 \ln x + x^2$

$\underline{y' = x^2(3\ln x + 1)}$

f) $y = \frac{\ln x}{e^{2x}}$

$u = \ln x$

$u' = \frac{1}{x}$

$v = e^{2x}$

$v' = 2e^{2x}$

$y' = \frac{u'v - uv'}{v^2}$

$y' = \frac{\frac{e^{2x}}{x} - 2e^{2x}\ln x}{\left(e^{2x}\right)^2}$

$y' = \frac{\frac{e^{2x} - 2xe^{2x}\ln x}{x}}{\left(e^{2x}\right)^2}$

$y' = \frac{e^{2x}\left(1 - 2x\ln x\right)}{x\left(e^{2x}\right)^2}$

$\underline{y' = \frac{1 - 2x\ln x}{xe^{2x}}}$

g) $y = \ln\left(x^4 e^{2x}\right)$

$y = \ln x^4 + \ln e^{2x}$

$y = 4\ln x + 2x$

$\underline{y' = \frac{4}{x} + 2}$

h) $y = \ln\left(\frac{x}{2x+1}\right)$

$y = \ln x - \ln\left(2x+1\right)$

$y' = \frac{1}{x} - \frac{2}{2x+1}$

$y' = \frac{2x+1-2x}{x(2x+1)}$

$\underline{y' = \frac{1}{x(2x+1)}}$

i) $y = \ln\sqrt{2x^3+1}$

$y = \ln\left(2x^3+1\right)^{\frac{1}{2}}$

$y = \frac{1}{2}\ln\left(2x^3+1\right)$

$y' = \frac{1}{2}\left(6x^2\right)\frac{1}{2x^3+1}$

$\underline{y' = \frac{3x^2}{2x^3+1}}$

Scholar Page 63 Q52 - 61

Worksheet – Differentiation of Log Functions

1. Use the chain rule to differentiate each function with respect to x and simplify where possible.

a) $y=\ln(2x+1)$ b) $y=\ln(4x+1)$ c) $y=\ln(x^2+1)$ d) $y=\ln(x-1)$
e) $y=\ln(x^4+1)$ f) $y=\ln(\cos x)$ g) $y=\ln(2x^2+1)$ h) $y=\ln(1-2x)$
i) $y=\ln(\sin x)$ j) $y=\ln(3x+2)$ k) $y=\ln(2x^3+1)$ l) $y=\ln(\cos 2x)$
m) $y=\ln(e^x+1)$ n) $y=\ln(\ln x)$ o) $y=\ln(\sec x)$ p) $y=\ln(\sec x+\tan x)$
q) $y=(\ln x)^2$ r) $y=\ln(\sin 4x)$

2. Use the product rule or quotient rule to differentiate each function with respect to x and simplify.

a) $y=x\ln x$ b) $y=e^x\ln x$ c) $y=x^2\ln x$ d) $y=\dfrac{\ln x}{x}$

e) $y=\dfrac{x}{\ln x}$ f) $y=\dfrac{\ln x}{e^x}$ g) $y=\dfrac{x^2}{\ln x}$ h) $y=\dfrac{\ln x}{x^3}$

i) $y=x^4\ln x$ j) $y=e^{2x}\ln x$ k) $y=\dfrac{e^x}{\ln x}$

3. Using the laws of logarithms first, differentiate each function with respect to x and simplify where possible.

a) $y=\ln(x^2e^x)$ b) $y=\ln\sqrt{2x+1}$ c) $y=\ln\left(\dfrac{x}{x+1}\right)$ d) $y=\ln(x^3e^{2x})$

e) $y=\ln\sqrt{x^2+1}$ f) $y=\ln\left(\dfrac{x^2}{2x+1}\right)$ g) $y=\ln(x\cos x)$ h) $y=\ln\sqrt{x^4+1}$

i) $y=\ln\left(\dfrac{e^{2x}}{x^4}\right)$

4.a) Given $y=\ln\left(\dfrac{x-1}{x+1}\right)$, show that $\dfrac{dy}{dx}=\dfrac{2}{x^2-1}$.

b) Given $y=\ln\left(\dfrac{2x-1}{2x+1}\right)$, show that $\dfrac{dy}{dx}=\dfrac{4}{4x^2-1}$.

5. By taking the natural log of both sides, differentiate the following with respect to x.

a) $y=2^x$ b) $y=10^x$ c) $y=4^x$

6. Differentiate the following with respect to x using the following method:

$$y=\log_{10}x$$
$$10^y=x$$
$$\ln(10^y)=\ln x$$

a) $y=\log_2 x$ b) $y=\log_3 x$ c) $y=\log_5 x$

ANSWERS

1.
a) $\frac{2}{2x+1}$ b) $\frac{4}{4x+1}$ c) $\frac{2x}{x^2+1}$ d) $\frac{1}{x-1}$ e) $\frac{4x^3}{x^4+1}$ f) $-\tan x$ g) $\frac{4x}{2x^2+1}$

h) $-\frac{2}{1-2x}$ i) $\cot x$ j) $\frac{3}{3x+2}$ k) $\frac{6x^2}{2x^3+1}$ l) $-2\tan 2x$ m) $\frac{e^x}{e^x+1}$ n) $\frac{1}{x\ln x}$

o) $\tan x$ p) $\sec x$ q) $\frac{2\ln x}{x}$ r) $4\cot 4x$

2.
a) $1+\ln x$ b) $\frac{e^x(1+x\ln x)}{x}$ c) $x(1+2\ln x)$ d) $\frac{1-\ln x}{x^2}$ e) $\frac{\ln x-1}{(\ln x)^2}$

f) $\frac{1-x\ln x}{xe^x}$ g) $\frac{x(2\ln x-1)}{(\ln x)^2}$ h) $\frac{1-3\ln x}{x^4}$ i) $x^3(1+4\ln x)$ j) $\frac{e^{2x}(1+2x\ln x)}{x}$ k) $\frac{e^x(x\ln x-1)}{x(\ln x)^2}$

3.
a) $\frac{2}{x}+1$ b) $\frac{1}{2x+1}$ c) $\frac{1}{x(x+1)}$ d) $\frac{3}{x}+2$ e) $\frac{x}{x^2+1}$

f) $\frac{2(x+1)}{x(2x+1)}$ g) $\frac{1}{x}-\tan x$ h) $\frac{2x^3}{x^4+1}$ i) $2-\frac{4}{x}$

4.a) Proofs
b) Proofs

5.a) $2^x\times\ln 2$ b) $10^x\times\ln 10$ c) $2\times 4^{2x}\times\ln 4$

6.a) $\frac{1}{x\ln 2}$ b) $\frac{1}{x\ln 3}$ c) $\frac{1}{x\ln 5}$

Stationary Points and Points of Inflection

The second derivative is used to investigate the nature of stationary points on the curve $y = f(x)$

$$\frac{d^2y}{dx^2} > 0 \quad \text{at stationary points} \Rightarrow \text{Minimum Turning Point}$$
$$\frac{d^2y}{dx^2} < 0 \quad \text{at stationary points} \Rightarrow \text{Maximum Turning Point}$$
$$\frac{d^2y}{dx^2} = 0 \quad \text{gives NO INFORMATION}$$

If the second derivative cannot be easily found, use a nature table.

Ex1. Determine the coordinate and nature of the stationary points on the curve of $y = 2x^3 - 3x^2 - 12x$

$$y = 2x^3 - 3x^2 - 12x$$
$$\frac{dy}{dx} = 6x^2 - 6x - 12$$
$$\frac{dy}{dx} = 6(x^2 - x - 2)$$

SP when $\frac{dy}{dx} = 0$

$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2, x = -1$$

Sub values into $y = 2x^3 - 3x^2 - 12x$

$x = 2, y = -20 \quad (2, -20)$

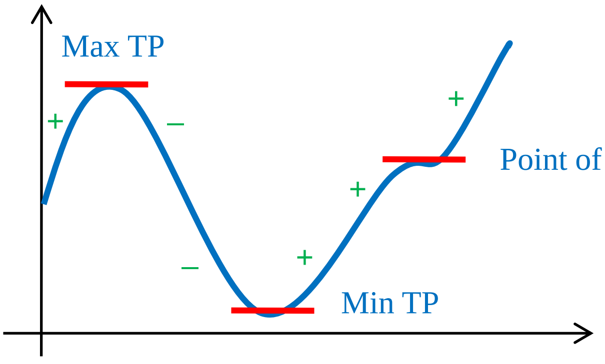
$x = -1, y = 7 \quad (-1, 7)$

Nature - $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = 12x - 6$$
$$x = 2, \frac{d^2y}{dx^2} = 24 - 6 = 18 \quad > 0, \ \text{Minimum}$$
$$x = -1, \frac{d^2y}{dx^2} = -12 - 6 = -18 \quad < 0, \ \text{Maximum}$$

Maximum at $(-1, 7)$
Minimum at $(2, -20)$

Point of Inflection



To find the Stationary Point – find x values such that : $\frac{dy}{dx}=0$

Get co-ordinate - use the x values to find the y value .

To find the nature of the points, substitute x values into $\frac{d^2y}{dx^2}$ and

$\frac{d^2y}{dx^2}<0$ MAXIMUM T.P.

$\frac{d^2y}{dx^2}>0$ MINIMUM T.P.

$\frac{d^2y}{dx^2}=0$ NATURE TABLE TO DETERMINE POI

Ex. 1
Find the coordinates of the points of inflection on the curve

$$y=x^6-3x^4+3x^2$$
$$\frac{dy}{dx}=6x^5-12x^2+6x$$
$$\frac{dy}{dx}=6x\left(x^4-2x^2+1\right)$$

SP occurs when $\frac{dy}{dx}=0$

$$6x\left(x^2-1\right)\left(x^2-1\right)=0$$
$$\underline{\underline{x=0 \quad x=-1 \quad x=1}}$$

Nature

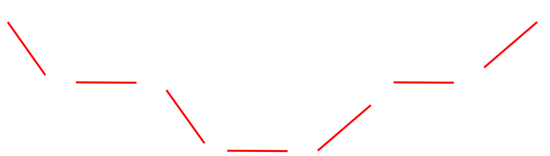
$$\frac{d^2y}{dx^2}=30x^4-36x^2+6$$

$x=0$ $\frac{d^2y}{dx^2}=6$ >0 $\therefore$ Min

$x=-1$ $\frac{d^2y}{dx^2}=0$ POI?

$x=1$ $\frac{d^2y}{dx^2}=0$ POI?

	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
$\frac{dy}{dx}=6x^5-12x^2+6x$	-108	0	-1.69	0	1.69	0	1.08



$y=x^6-3x^4+3x^2$
$x=-1$ $y=1-3+3=1$
$x=1$ $y=1-3+3=1$

Points of Inflection

(-1, 1) (1, 1)

Point of Inflection

Ex2. Show that there are no points of inflection on the curve $y = \frac{x-3}{x+2}$.

$$y = \frac{x-3}{x+2}$$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

$$\frac{dy}{dx} = \frac{x+2-(x-3)}{(x+2)^2}$$

$$\frac{dy}{dx} = \frac{5}{(x+2)^2}$$

$$\frac{d^2y}{dx^2} = -10(x+2)^{-3} = -\frac{10}{(x+2)^3}$$

Points of Inflection occur when $\frac{d^2y}{dx^2} = 0$

$$-\frac{10}{(x+2)^3} = 0$$

$$-10 = 0\left(x+2\right)^3$$

$-10 = 0$ this is not possible therefore there is NO point of inflection.

Scholar Page 51 Q16 - 17

Worksheet – Higher Derivatives

1. Find the second derivative $\frac{d^2y}{dx^2}$ in each case.

a) $y = x^4$ b) $y = 2x^6$ c) $y = 4x^2$ d) $y = 3x^5$

e) $y = \sin x$ f) $y = e^{3x}$ g) $y = 3x^4 + 2x^3 + 4x^2 + 5x + 1$

h) $y = x^3 - 2x^2 + 6x + 4$ i) $y = \sin 2x$ j) $y = (x+1)^5$

k) $y = (2x+1)^6$ l) $y = (3x-1)^4$ m) $y = 3e^{2x} + 4e^{-2x}$ n) $y = 4\cos 2x$

o) $y = 3x^3 - 6x + 4$ p) $y = (4x+1)^{\frac{3}{2}}$ q) $y = \sqrt{2x-1}$ r) $y = \ln x$

s) $y = \ln(\cos x)$

2. Given $y = (x^2+1)^4$, show that $\frac{d^2y}{dx^2} = 8(7x^2+1)(x^2+1)^2$.

3. Given $y = e^x \sin x$, show that $\frac{d^2y}{dx^2} = 2e^x \cos x$.

4. Given $y = \sin(x^2)$, show that $\frac{d^2y}{dx^2} = 2[\cos(x^2) - 2x^2 \sin(x^2)]$.

5. Given $y = x^2 \ln x$, show that $\frac{d^2y}{dx^2} = 3 + 2\ln x$.

6. Given $y = e^{x^2}$, show that $\frac{d^2y}{dx^2} = 2(2x^2+1)e^{x^2}$.

7. Given $y = x\sin x$, show that $\frac{d^2y}{dx^2} = 2\cos x - x\sin x$.

8. Given $y = \frac{x}{2x+1}$, show that $\frac{d^2y}{dx^2} = -\frac{4}{(2x+1)^3}$.

9. Given $y = e^{ax}$, where a is a constant, find expressions for $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

   Conjecture an expression for the n[th] derivative $\frac{d^ny}{dx^n}$.

10. Given $y = xe^x$ show that $\frac{dy}{dx} = (x+1)e^x$ and obtain similar expressions for $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

    Conjecture an expression for the n[th] derivative $\frac{d^ny}{dx^n}$.

11. Given $y = x^2e^x$, show that $\frac{d^2y}{dx^2} = (x^2+4x+2)e^x$.

ANSWERS

1.

a) $12x^2$ b) $60x^4$ c) 8 d) $60x^3$ e) $-\sin x$ f) $9e^{3x}$

g) $36x^2+12x+8$ h) $6x-4$ i) $-4\sin 2x$ j) $20(x+1)^3$ k) $120(2x+1)^4$

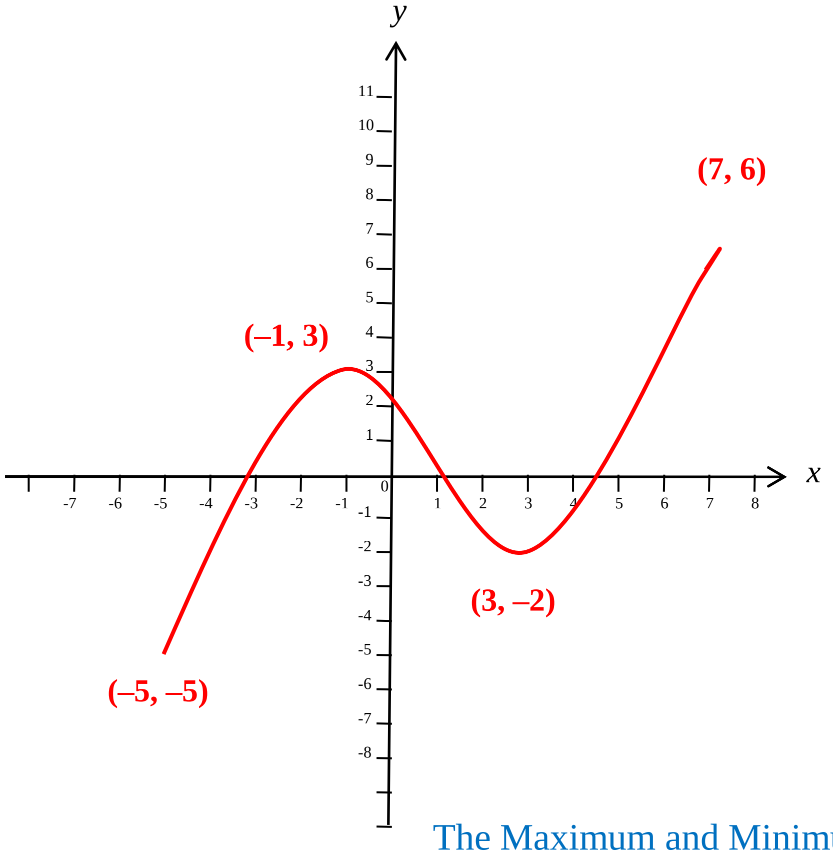
l) $108(3x-1)^2$ m) $12e^{2x}+16e^{-2x}$ n) $-16\cos 2x$ o) $18x$

p) $\dfrac{12}{\sqrt{4x+1}}$ q) $-\dfrac{1}{(2x-1)^{3/2}}$ r) $-\dfrac{1}{x^2}$ s) $-\sec^2 x$

9. $\dfrac{dy}{dx}=ae^{ax}$, $\dfrac{d^2y}{dx^2}=a^2e^{ax}$, $\dfrac{d^3y}{dx^3}=a^3e^{ax}$, $\dfrac{d^ny}{dx^n}=a^ne^{ax}$.

10. $\dfrac{d^2y}{dx^2}=(x+2)e^x$, $\dfrac{d^3y}{dx^3}=(x+3)e^x$, $\dfrac{d^ny}{dx^n}=(x+n)e^x$.

Curve Sketching on a Closed Interval



$-5 \le x \le 7$ *Max* of 6, *Min* of -5

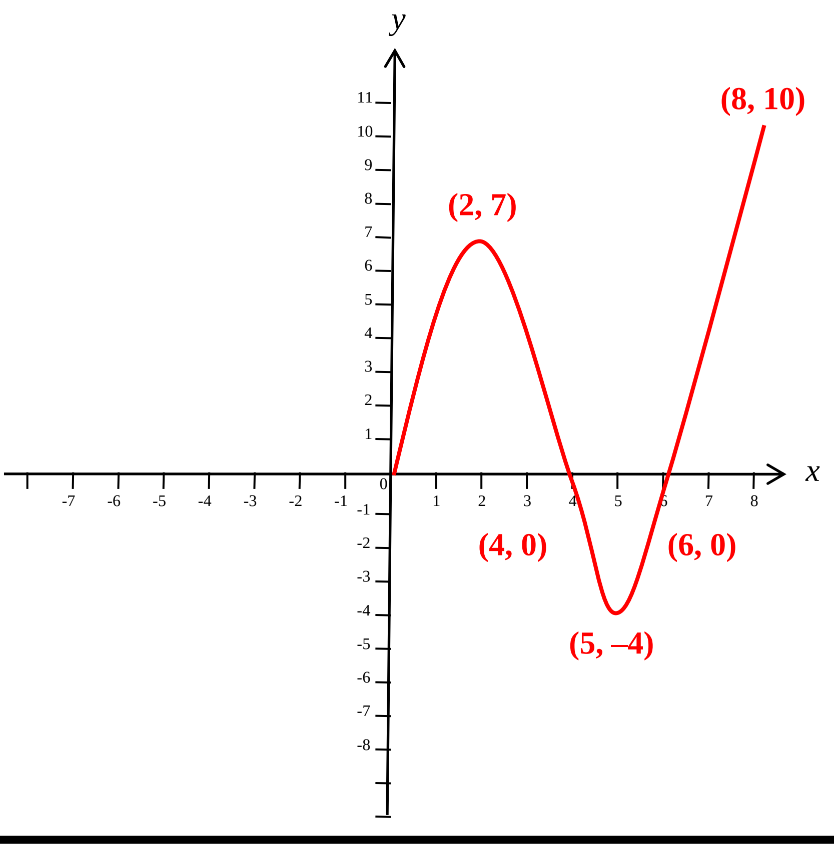
$0 \le x \le 4$ *Max* of -2, *Min* of -2

$-3 \le x \le 4$ *Max* of 3, *Min* of -2

The Maximum and Minimum values can occur at either

- Stationary Points or
- End Points

Q62 - Scholar



$a)\, 0 \le x \le 8$ *Max* of 10, *Min* of -4

$b)\, 0 \le x \le 4$ *Max* of 7, *Min* of 0

$c)\, 2 \le x \le 6$ *Max* of 7, *Min* of 4

Scholar Page65 Q63 - 71

Differentiation of Inverse Functions

Unit 1 Outcome 4 covered finding the inverse of a function, which graphically is the reflection of the function under the line $y = x$.

The inverse of $y = f(x)$ becomes $f^{-1}(y) = x$. If the function is $y = f(x)$ the inverse is $x = f(y)$.

To differentiate an inverse function, $\dfrac{dy}{dx} = \dfrac{1}{dx/dy}$

Example: Given $f(x) = x^3$ find $f'(x)$ and state the derivative of $f^{-1}(x)$.

$$f(x) = x^3 \quad \rightarrow \quad y = x^3$$
$$f^{-1}(x) = f(y) \quad \rightarrow \quad x = y^3$$
$$f^{-1}(x) = \quad \rightarrow \quad y = x^{1/3}$$
$$f'(y) = 3y^2$$
$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(y)} = \frac{1}{3y^2}$$
$$y = x^{1/3}$$
$$\frac{d}{dx} f^{-1}(x) = \frac{1}{3\left(x^{1/3}\right)^2} = \frac{1}{3x^{2/3}}$$

Let $y = f^{-1}(x)$ and therefore $x = f(y)$

Find $f'(y) = dx/dy$ by differentiating with respect to y.

Use the formula $\dfrac{dy}{dx} = \dfrac{1}{dx/dy}$.

Substitute y (see previous $f^{-1}(x)$ where $y = \ldots$).

Quick method: $f^{-1}(x) = f(y) \quad \rightarrow \quad y = x^{1/3}$
$$\frac{d}{dx} f^{-1}(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

Examples

1) $f(x) = x^5$

$$f(x) = y = x^5$$
$$f(y) = x = y^5$$
$$\underline{y = x^{1/5}}$$

Method 1

$$y = x^{1/5}$$
$$\frac{dy}{dx} = \frac{1}{5}x^{-4/5}$$
$$\frac{dy}{dx} = \frac{1}{5x^{4/5}}$$

Method 2

$$x = y^5$$
$$\frac{dx}{dy} = 5y^4$$
$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{5y^4}$$
$$y = x^{1/5}$$
$$\frac{dy}{dx} = \frac{1}{5x^{4/5}}$$

2) $f(x) = x^2 + 1, x > 1$

$$y = x^2 + 1$$
$$f(y) \quad x = y^2 + 1$$
$$y^2 = x - 1$$
$$\underline{y = (x-1)^{\frac{1}{2}}}$$
$$\frac{dy}{dx} = \frac{1}{2}(x-1)^{-1/2}$$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{x-1}}$$

Scholar p6 Q5 - 8

Differentiation of Inverse Trigonometric Functions

$f(x) = \sin x$

$f^{-1}(x) = \sin^{-1} x$ **Also known as arcsin.**

$f(x) = \sin x$	$f^{-1}(x) = \sin^{-1} x$
$f(x) = \cos x$	$f^{-1}(x) = \cos^{-1} x$
$f(x) = \tan x$	$f^{-1}(x) = \tan^{-1} x$

eg $\sin\frac{\pi}{6} = \frac{1}{2}$, therefore $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

To Differentiate an Inverse Trig function (PROOF):

$y = \sin^{-1} x$ where $-1 < x < 1$.

$\sin y = \sin\left(\sin^{-1} x\right)$ **Find sin of both sides to eliminate sin⁻¹.**

$\sin y = x$

$x = \sin y$

$\frac{dx}{dy} = \cos y$ **Differentiate *x* = sin *y* with respect to *y*.**

$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\cos y}$. **Replace cos *y* in terms of *x*.**

$\sin^2 y + \cos^2 y = 1$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \sqrt{1 - \sin^2 y}$ **Replace sin *y* = *x***

$\cos y = \sqrt{1 - x^2}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$ **Substitute into derivative.**

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$

Examples of Differentiating Inverse Trig Functions

1) $y=\sin^{-1}(3x)$ ***(Use Chain Rule)***

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-(3x)^2}}(3)$$

$$\frac{dy}{dx}=\frac{3}{\sqrt{1-9x^2}}$$

2) $y=\tan^{-1}\left(x^2\right)$

$$\frac{dy}{dx}=\frac{1}{1+\left(x^2\right)^2}(2x)$$

$$\frac{dy}{dx}=\frac{2x}{1+x^4}$$

3) $y=\cos^{-1}(1-2x)$

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-(1-2x)^2}}(-2)$$

$$\frac{dy}{dx}=\frac{2}{\sqrt{1-\left(1-4x+4x^2\right)}}$$

$$\frac{dy}{dx}=\frac{2}{\sqrt{4x-4x^2}}$$

$$\frac{dy}{dx}=\frac{2}{\sqrt{4}\sqrt{x-x^2}}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{x-x^2}}$$

4) $y=\tan^{-1}\left(\sqrt{4x-1}\right), x>\frac{1}{4}$

$$u=(4x-1)^{1/2}$$

$$\frac{du}{dx}=\frac{1}{2}(4x-1)^{-1/2}(4)$$

$$\frac{du}{dx}=\frac{2}{\sqrt{4x-1}}$$

$$\frac{dy}{dx}=\frac{1}{1+\left(\sqrt{4x-1}\right)^2}\left(\frac{2}{\sqrt{4x-1}}\right)$$

$$\frac{dy}{dx}=\frac{2}{(1+4x-1)\sqrt{4x-1}}$$

$$\frac{dy}{dx}=\frac{2}{4x\sqrt{4x-1}}$$

$$\frac{dy}{dx}=\frac{1}{2x\sqrt{4x-1}}$$

Examples of Differentiating Inverse Trig Functions

5) $y = \sin^{-1}\left(\dfrac{x}{3}\right)$

$$u = \frac{1}{3}x$$

$$\frac{du}{dx} = \frac{1}{3}$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}}\left(\frac{1}{3}\right)$$

$$\frac{dy}{dx} = \frac{1}{3\sqrt{1-\frac{x^2}{9}}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{9}\sqrt{1-\frac{x^2}{9}}}$$

$$\underline{\underline{\frac{dy}{dx} = \frac{1}{\sqrt{9-x^2}}}}$$

6) $y = x^2 \sin^{-1} x$

$u = x^2 \qquad u' = 2x$

$v = \sin^{-1} x \qquad v' = \dfrac{1}{\sqrt{1-x^2}}$

$$\frac{dy}{dx} = u'v + uv'$$

$$\underline{\underline{\frac{dy}{dx} = 2x\sin^{-1} x + \frac{x^2}{\sqrt{1-x^2}}}}$$

7) $y = \sqrt{1-x^2}\cos^{-1} x$

$u = \left(1-x^2\right)^{\frac{1}{2}} \qquad u' = \dfrac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x) \qquad u' = -\dfrac{x}{\sqrt{1-x^2}}$

$v = \cos^{-1} x \qquad v' = -\dfrac{1}{\sqrt{1-x^2}}$

$$\frac{dy}{dx} = u'v + uv'$$

$$\frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}\left(\cos^{-1} x\right) - \frac{1}{\sqrt{1-x^2}}\left(\sqrt{1-x^2}\right)$$

$$\underline{\underline{\frac{dy}{dx} = -\frac{x\cos^{-1} x}{\sqrt{1-x^2}} - 1}}$$

$f(x)$	$f'(x)$
$\sin^{-1}\left(\dfrac{x}{a}\right)$	$\dfrac{1}{\sqrt{a^2-x^2}}$
$\cos^{-1}\left(\dfrac{x}{a}\right)$	$-\dfrac{1}{\sqrt{a^2-x^2}}$
$\tan^{-1}\left(\dfrac{x}{a}\right)$	$\dfrac{a}{a^2+x^2}$

Worksheet: Differentiation of Inverse Trig 1 **Scholar p8 Q11 - 22**

$$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$$

1. Use the chain rule to differentiate each function with respect to x and simplify where possible.

a) $y=\sin^{-1}(2x)$ b) $y=\cos^{-1}(3x)$ c) $y=\tan^{-1}(4x)$ d) $y=\sin^{-1}(x^2)$

e) $y=\tan^{-1}(x^3)$ f) $y=\cos^{-1}(6x)$ g) $y=\tan^{-1}(x+2)$ h) $y=\sin^{-1}\left(\sqrt{x}\right)$

i) $y=\tan^{-1}\left(\sqrt{x}\right)$ j) $y=\tan^{-1}(e^x)$ k) $y=\sin^{-1}(x-1)$ l) $y=\tan^{-1}(2x+1)$

m) $y=\tan^{-1}\left(\sqrt{x-1}\right)$ n) $y=\cos^{-1}(2x-1)$ o) $y=\sin^{-1}(e^x)$ p) $y=\tan^{-1}(e^{2x})$

q) $y=\sin^{-1}\left(\frac{x}{2}\right)$ r) $y=\sin^{-1}\left(\frac{1}{x}\right)$ s) $y=\tan^{-1}\left(\sqrt{2x-1}\right)$ t) $y=\sin^{-1}(\cos x)$

2. Given $y=\sin^{-1}(\sin x)$, show that $\frac{dy}{dx}=1$ and explain this answer.

3. a) Show that $\frac{d}{dx}\left(\frac{x-1}{x+1}\right)=\frac{2}{(x+1)^2}$.

b) Hence, given $y=\tan^{-1}\left(\frac{x-1}{x+1}\right)$, show that $\frac{dy}{dx}=\frac{1}{x^2+1}$.

4. a) Show that $\frac{d}{dx}\left(\frac{e^x}{x}\right)=\frac{(x-1)e^x}{x^2}$.

b) Hence, given $y=\tan^{-1}\left(\frac{e^x}{x}\right)$, show that $\frac{dy}{dx}=\frac{(x-1)e^x}{x^2+e^{2x}}$.

5. a) Show that $\frac{d}{dx}\left(\frac{\ln x}{x}\right)=\frac{1-\ln x}{x^2}$.

b) Hence, given $y=\tan^{-1}\left(\frac{\ln x}{x}\right)$, show that $\frac{dy}{dx}=\frac{1-\ln x}{x^2+(\ln x)^2}$.

6. Use the product rule to differentiate each function with respect to x.

a) $y=x\sin^{-1}x$ b) $y=x\tan^{-1}x$ c) $y=x^2\cos^{-1}x$

d) $y=x^3\sin^{-1}x$ e) $y=\left(1+x^2\right)\tan^{-1}x$ f) $y=e^x\cos^{-1}x$

g) $y=\ln x\tan^{-1}x$ h) $y=e^{2x}\sin^{-1}x$ i) $y=\sqrt{1-x^2}\sin^{-1}x$

ANSWERS

1.

a) $\frac{2}{\sqrt{1-4x^2}}$ b) $-\frac{3}{\sqrt{1-9x^2}}$ c) $\frac{4}{1+16x^2}$ d) $\frac{2x}{\sqrt{1-x^4}}$

e) $\frac{3x^2}{1+x^6}$ f) $-\frac{6}{\sqrt{1-36x^2}}$ g) $\frac{1}{x^2+4x+5}$ h) $\frac{1}{2\sqrt{x-x^2}}$

i) $\frac{1}{2\sqrt{x(1+x)}}$ j) $\frac{e^x}{1+e^{2x}}$ k) $\frac{1}{\sqrt{2x-x^2}}$ l) $\frac{1}{2x^2+2x+1}$

m) $\frac{1}{2x\sqrt{x-1}}$ n) $-\frac{1}{\sqrt{x-x^2}}$ o) $\frac{e^x}{\sqrt{1-e^{2x}}}$ p) $\frac{2e^{2x}}{1+e^{4x}}$

q) $\frac{1}{\sqrt{4-x^2}}$ r) $-\frac{1}{x\sqrt{x^2-1}}$ s) $\frac{1}{2x\sqrt{2x-1}}$ t) -1

2. $y = \sin^{-1}(\sin x) = x$, so $\frac{dy}{dx} = 1$

3.

a) $\frac{x}{\sqrt{1-x^2}} + \sin^{-1} x$ b) $\frac{x}{1+x^2} + \tan^{-1} x$ c) $2x\cos^{-1} x - \frac{x^2}{\sqrt{1-x^2}}$

d) $\frac{x^3}{\sqrt{1-x^2}} + 3x^2 \sin^{-1} x$ e) $1 + 2x\tan^{-1} x$ f) $e^x \cos^{-1} x - \frac{e^x}{\sqrt{1-x^2}}$

g) $\frac{\ln x}{1+x^2} + \frac{\tan^{-1} x}{x}$ h) $\frac{e^{2x}}{\sqrt{1-x^2}} + 2e^{2x}\sin^{-1} x$ i) $1 - \frac{x\sin^{-1} x}{\sqrt{1-x^2}}$

Implicit Differentiation

An equation expressed as $y = 3x + 2$, y is expressed **explicitly** as a function of x.

The equation $x^2 + y^2 = 1$, defines y **implicitly** in terms of x.

By rearranging this equation, $y^2 = 1 - x^2$ therefore $y = \pm\sqrt{(1-x^2)}$.

This gives **two** explicit functions of x and is called a *multiple valued function*.

It is not always easy to rearrange an equation, expressing y explicitly in order to differentiate.

It is possible to differentiate an implicit equation.

When differentiating y^2 with respect to x, the **chain rule** must be used. $\frac{d}{dx}\left(y^2\right) = 2y.\frac{dy}{dx}$

Examples (all with respect to x) :

1) Differentiate y^3: $\frac{d}{dx}\left(y^3\right) = 3y^2.\frac{dy}{dx}$

2) Differentiate $\sin y$: $\frac{d}{dx}\left(\sin y\right) = \cos y.\frac{dy}{dx}$

3) Differentiate e^y: $\frac{d}{dx}\left(e^y\right) = e^y.\frac{dy}{dx}$

4) Differentiate $\ln y$: $\frac{d}{dx}\left(\ln y\right) = \frac{1}{y}.\frac{dy}{dx}$

5) Differentiate xy: $\frac{d}{dx}\left(xy\right) = u'v + uv'$

$u = x \quad u' = 1$

$v = y \quad y' = 1.\frac{dy}{dx}$

MUST USE PRODUCT RULE.
x and y are separate functions.

$$\frac{d}{dx}\left(xy\right) = 1.y + x\frac{dy}{dx}$$

$$\frac{d}{dx}\left(xy\right) = y + x\frac{dy}{dx}$$

Implicit Differentiation - Examples

1) The equation $x^2 + y^2 = 4$ defines y implicitly in terms of x. Find an expression for $\frac{dy}{dx}$ in terms of x and y.

$$x^2 + y^2 = 4$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} = -x$$

$$\underline{\underline{\frac{dy}{dx} = -\frac{x}{y}}}$$

Differentiate both sides with respect to x.

2) The equation $\ln y = x + y$ defines y implicitly in terms of x. Find an expression for $\frac{dy}{dx}$ in terms of x and y.

$$\ln y = x + y$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} - \frac{dy}{dx} = 1$$

$$\frac{dy}{dx}\left(\frac{1}{y} - 1\right) = 1$$

$$\frac{dy}{dx}\left(\frac{1}{y} - \frac{y}{y}\right) = 1$$

$$\frac{dy}{dx}\left(\frac{1-y}{y}\right) = 1$$

$$\underline{\underline{\frac{dy}{dx} = \frac{y}{1-y}}}$$

3) The equation $y^2 - xy = x$ defines y implicitly in terms of x. Find an expression for $\frac{dy}{dx}$ in terms of x and y.

$$y^2 - xy = x$$

$$2y\frac{dy}{dx} + u'v + uv' = 1$$

$$2y\frac{dy}{dx} - y - x\frac{dy}{dx} = 1$$

$$\frac{dy}{dx}(2y - x) = 1 + y$$

$$\underline{\underline{\frac{dy}{dx} = \frac{1+y}{2y-x}}}$$

$$u = -x \qquad u' = -1$$

$$v = y \qquad v' = \frac{dy}{dx}$$

Worksheet: Implicit Differentiation 1 – Q1 & 2 **Scholar p12 Q30 - 41**

1. y is a function of x defined implicitly by each equation below. Find an expression for in terms of x and y in each case.

a) $x^2 + y^2 = 1$ b) $x^2 - y^2 = 4$ c) $x^2 + 2y^2 = 9$ d) $x^2 - \ln y = 0$

e) $x^2 + e^y = 1$ f) $\sin x + e^y = 0$ g) $y^2 = 2x + 2y$ h) $\sin x + \sin y = 0$

i) $y^2 - 3x = 4x^2$ j) $2y^2 - x^2 = 1$ k) $3x^2 + y^2 = 5x$ l) $3y = 2x^3 + \cos y$

m) $\ln(x+y) = x$ n) $\sin x + 2\cos y = 1$ o) $x^4 = x^2 - y^2$

p) $2x^2 + 2y^2 - 4x + 4y - 9 = 0$

2. y is a function of x defined implicitly by each equation below. Find an expression for in terms of x and y in each case.

a) $xy = 4$ b) $x^2 + xy = 1$ c) $x^3 + 1 = xy$ d) $xy = x^2 + y^2$

e) $x^2 + xy + y^2 = 0$ f) $x^2 + 4xy + y^2 = 8$ g) $x^2 - xy + 3y^2 = 10$ h) $xy^2 = 1$

i) $xe^y = 1$ j) $x^2 y = 9$ k) $xy = x - y$ l) $xy + y^2 = 2$

m) $y^3 + 2xy = x^2 + 1$ n) $x^3 + xy = 1 - y$ o) $x \ln y = x^2 + y^2$ p) $x \sin y + y = \cos x$

q) $xy^2 + 1 = y$ r) $x^3 - xy + y^2 = 1$

3. y is defined implicitly in terms of x by the equation $x^2 + y^2 = 25$.

a) Find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

b) Find the equation of the tangent at the point $(3,-4)$ on the curve.

4. y is defined implicitly in terms of x by the equation $x^2 - y^2 = 9$.

a) Find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

b) Find the equation of the tangent at the point $(5,4)$ on the curve.

5. y is defined implicitly in terms of x by the equation $3x^2 + 5y^2 = 17$.

a) Find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

b) Find the equation of the tangent at the point $(-2,1)$ on the curve.

6. y is defined implicitly in terms of x by the equation $xy = 4$.

a) Find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

b) Find the equation of the tangent at the point $(1,4)$ on the curve.

ANSWERS

1.

a) $\frac{dy}{dx} = -\frac{x}{y}$ b) $\frac{dy}{dx} = \frac{x}{y}$ c) $\frac{dy}{dx} = -\frac{x}{2y}$ d) $\frac{dy}{dx} = 2xy$

e) $\frac{dy}{dx} = -\frac{2x}{e^y}$ f) $\frac{dy}{dx} = -\frac{\cos x}{e^y}$ g) $\frac{dy}{dx} = \frac{1}{y-1}$ h) $\frac{dy}{dx} = -\frac{\cos x}{\cos y}$

i) $\frac{dy}{dx} = \frac{8x+3}{2y}$ j) $\frac{dy}{dx} = \frac{x}{2y}$ k) $\frac{dy}{dx} = \frac{5-6x}{2y}$ l) $\frac{dy}{dx} = \frac{6x^2}{3+\sin y}$

m) $\frac{dy}{dx} = x+y-1$ n) $\frac{dy}{dx} = \frac{\cos x}{2\sin y}$ o) $\frac{dy}{dx} = \frac{x(1-2x^2)}{y}$ p) $\frac{dy}{dx} = \frac{1-x}{1+y}$

2.

a) $\frac{dy}{dx} = -\frac{y}{x}$ b) $\frac{dy}{dx} = -\frac{(2x+y)}{x}$ c) $\frac{dy}{dx} = \frac{3x^2-y}{x}$ d) $\frac{dy}{dx} = \frac{2x-y}{x-2y}$

e) $\frac{dy}{dx} = -\frac{(2x+y)}{(x+2y)}$ f) $\frac{dy}{dx} = -\frac{(x+2y)}{(2x+y)}$ g) $\frac{dy}{dx} = \frac{y-2x}{6y-x}$ h) $\frac{dy}{dx} = -\frac{y}{2x}$

i) $\frac{dy}{dx} = -\frac{1}{x}$ j) $\frac{dy}{dx} = -\frac{2y}{x}$ k) $\frac{dy}{dx} = \frac{1-y}{1+x}$ l) $\frac{dy}{dx} = -\frac{y}{(x+2y)}$

m) $\frac{dy}{dx} = \frac{2(x-y)}{2x+3y^2}$ n) $\frac{dy}{dx} = -\frac{(3x^2+y)}{(x+1)}$ o) $\frac{dy}{dx} = \frac{y(2x-\ln y)}{x-2y^2}$ p) $\frac{dy}{dx} = -\frac{(\sin x+\sin y)}{(x\cos y+1)}$

q) $\frac{dy}{dx} = -\frac{y^2}{(2xy-1)} = \frac{y^2}{1-2xy}$ r) $\frac{dy}{dx} = \frac{y-3x^2}{2y-x}$

3.a) $\frac{dy}{dx} = -\frac{x}{y}$ b) $4y = 3x-25$

4.a) $\frac{dy}{dx} = \frac{x}{y}$ b) $4y = 5x-9$

5.a) $\frac{dy}{dx} = -\frac{3x}{5y}$ b) $5y = 6x+17$

6.a) $\frac{dy}{dx} = -\frac{y}{x}$ b) $y = -4x+8$

Implicit Differentiation – Equation of the Tangent

In order to find the equation of a tangent at a point

- differentiate the equation
- substitute the point into the differentiated equation to find the gradient
- substitute the gradient and point into the equation $y - b = m(x - a)$

1) A curve is defined by the implicit equation
Find the equation of the tangent at the point (3, 4) on the curve.

$$x^2 + y^2 + 2x - 4y = 15$$
$$2x + 2y\frac{dy}{dx} + 2 - 4\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(2y - 4) = -2x - 2$$
$$\frac{dy}{dx} = \frac{-2x-2}{2y-4}$$
$$\underline{\underline{\frac{dy}{dx} = \frac{-x-1}{y-2}}}$$

$$x = 3 \qquad y = 4$$
$$\frac{dy}{dx} = \frac{-3-1}{4-2}$$
$$\underline{\underline{\frac{dy}{dx} = -2}}$$

2) A curve is defined by the implicit equation $2x^2 - 3xy - y^2 = 1$
Find the equation of the tangent at the point (2, 1) on the curve.

$$u = -3x \qquad u' = -3$$
$$v = y \qquad v' = \frac{dy}{dx}$$

$$2x^2 - 3xy - y^2 = 1$$
$$4x + u'v + uv' - 2y\frac{dy}{dx} = 0$$
$$4x - 3y - 3x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(-3x - 2y) = 3y - 4x$$
$$\underline{\underline{\frac{dy}{dx} = \frac{3y-4x}{-3x-2y}}}$$

$$x = 2 \qquad y = 1$$
$$\frac{dy}{dx} = \frac{3-8}{-6-2}$$
$$\underline{\underline{\frac{dy}{dx} = \frac{5}{8}}}$$

$$y - b = m(x - a)$$
$$y - 1 = \frac{5}{8}(x - 2)$$
$$8y - 8 = 5x - 10$$
$$\underline{\underline{8y = 5x - 2}}$$

Worksheet: Implicit Differentiation 1 – Q3 -6
Implicit Differentiation 2

Scholar p15 Q44 - 50

1. y is defined implicitly in terms of x by the equation $x^2y = 9$.

   a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

   b) Find the equation of the tangent at the point $(3,1)$ on the curve.

2. y is defined implicitly in terms of x by the equation $x^2 + 4xy - y^2 = 16$.

   a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

   b) Find the equation of the tangent at the point $(2,2)$ on the curve.

3. y is defined implicitly in terms of x by the equation $x^2 - 4x + y^2 - 6y = 12$.

   a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

   b) Find the equation of the tangent at the point $(5,7)$ on the curve.

4. y is defined implicitly in terms of x by the equation $x^2y + xy^2 = 2$.

   a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

   b) Find the equation of the tangent at the point $(1,1)$ on the curve.

5. y is defined implicitly in terms of x by the equation $y^2 + 2xy = x^3 + 7$.

   a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

   b) Find the equation of the tangent at the point $(1,-4)$ on the curve.

6. y is defined implicitly in terms of x by the equation $xy + y^2 = 2$.

   a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

   b) Find the equation of the tangent at the point $(1,1)$ on the curve.

7. y is defined implicitly in terms of x by the equation $y^3 + 3xy = 3x^2 + 2$.

   a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

   b) Find the equation of the tangent at the point $(2,1)$ on the curve.

8. y is defined implicitly in terms of x by the equation $x^3 - 2y^3 = 3xy$.

   a) Find an expression for $\frac{dy}{dx}$ in terms of x and y.

   b) Find the equation of the tangent at the point $(2,1)$ on the curve.

ANSWERS

1.a) $\frac{dy}{dx}=-\frac{2y}{x}$ b) $3y=-2x+9$

2.a) $\frac{dy}{dx}=-\frac{(x+2y)}{(2x-y)}=\frac{2y+x}{y-2x}$ b) $y=-3x+8$

3.a) $\frac{dy}{dx}=\frac{2-x}{y-3}$ b) $4y=-3x+43$

4.a) $\frac{dy}{dx}=-\frac{y(2x+y)}{x(x+2y)}$ b) $y=-x+2$

5.a) $\frac{dy}{dx}=\frac{3x^2-2y}{2(x+y)}$ b) $6y=-11x-13$

6.a) $\frac{dy}{dx}=-\frac{y}{(x+2y)}$ b) $3y=-x+4$

7.a) $\frac{dy}{dx}=\frac{2x-y}{x+y^2}$ b) $y=x-1$

8.a) $\frac{dy}{dx}=\frac{x^2-y}{x+2y^2}$ b) $4y=3x-2$

Implicit Differentiation – Second Derivative

Worked Example - The function $y = f(x)$ is defined implicitly by the equation $x^2 + 2xy = 1$.

Find expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y.

Solution

$$x^2 + 2xy = 1$$

$$2x + u'v + uv' = 0$$

$u = 2x \quad u' = 2$

$v = y \quad v' = \frac{dy}{dx}$

$$2x + 2x\frac{dy}{dx} + 2y = 0$$

$(\div 2, \text{ and rearrange})$

$$\boxed{x + x\frac{dy}{dx} + y = 0}$$

$$x + x\frac{dy}{dx} + y = 0$$

$$x\frac{dy}{dx} = -x - y$$

$$\frac{dy}{dx} = \frac{-x-y}{x}$$

There are two possible methods to find the second derivative

Method 1:

Differentiate $x + x\frac{dy}{dx} + y = 0$ with respect to x.

$$x + x\frac{dy}{dx} + y = 0$$

$u = x \quad u' = 1$

$v = \frac{dy}{dx} \quad v' = \frac{d^2y}{dx^2}$

$$1 + u'v + uv' + \frac{dy}{dx} = 0$$

$$1 + x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$1 + x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0 \quad \left(\text{replace } \frac{dy}{dx} \text{ with } \frac{-x-y}{x}\right)$$

$$1 + x\frac{d^2y}{dx^2} + 2\frac{-x-y}{x} = 0 \quad (\text{mult by } x)$$

$$x + x^2\frac{d^2y}{dx^2} - 2x - 2y = 0$$

$$x^2\frac{d^2y}{dx^2} = x + 2y$$

$$\frac{d^2y}{dx^2} = \frac{x+2y}{x^2}$$

Method 2:

Differentiate $\frac{dy}{dx} = \frac{-x-y}{x}$ with respect to x using the quotient rule.

$u = -x - y \quad u' = -1 - \frac{dy}{dx}$

$v = x \quad v' = 1$

$$\frac{d^2y}{dx^2} = \frac{u'v - uv'}{v^2}$$

$$\frac{d^2y}{dx^2} = \frac{x\left(-1 - \frac{dy}{dx}\right) - (-x - y)}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{-x - x\frac{dy}{dx} + x + y}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{-x\frac{dy}{dx} + y}{x^2} \quad \left(\text{replace } \frac{dy}{dx} \text{ with } \frac{-x-y}{x}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-x\left(\frac{-x-y}{x}\right) + y}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{x + y + y}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{x+2y}{x^2}$$

Implicit Differentiation – Second Derivative

Example

The function $y = f(x)$ is defined implicitly by the equation $y^2 - x^2 = 4$.

Find expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y.

2 methods to work out 2nd derivative $\frac{d^2y}{dx^2}$

Method 1

$$y\frac{dy}{dx} - x = 0 \qquad u = y \quad u' = \frac{dy}{dx}$$

$$u'v + uv' - 1 = 0 \qquad v = \frac{dy}{dx} \quad v' = \frac{d^2y}{dx^2}$$

$$\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right) + y\frac{d^2y}{dx^2} - 1 = 0$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$\left(\frac{x}{y}\right)\left(\frac{x}{y}\right) + y\frac{d^2y}{dx^2} - 1 = 0$$

$$y\frac{d^2y}{dx^2} = 1 - \frac{x^2}{y^2}$$

$$y\frac{d^2y}{dx^2} = \frac{y^2}{y^2} - \frac{x^2}{y^2}$$

$$y\frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}$$

Method 2

$$\frac{dy}{dx} = \frac{x}{y} \qquad u = x \quad u' = 1$$

$$v = y \quad v' = \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{u'v - uv'}{v^2}$$

$$\frac{d^2y}{dx^2} = \frac{y - x\frac{dy}{dx}}{y^2}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$\frac{d^2y}{dx^2} = \frac{y - x\left(\frac{x}{y}\right)}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{y^2}{y} - \frac{x^2}{y}}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{y^2 - x^2}{y}}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}$$

Worksheet: Implicit Differentiation 3 **Scholar p18 Q51 - 54**

1. y is defined implicitly in terms of x by the equation $x^2 + y^2 = 4$.

Show that : a) $\dfrac{dy}{dx} = -\dfrac{x}{y}$ b) $\dfrac{d^2y}{dx^2} = -\dfrac{\left(x^2 + y^2\right)}{y^3}$

2. y is defined implicitly in terms of x by the equation $x^2 + xy = 3$.

Show that : a) $\dfrac{dy}{dx} = -\dfrac{(2x + y)}{x}$ b) $\dfrac{d^2y}{dx^2} = \dfrac{2(x + y)}{x^2}$

3. y is defined implicitly in terms of x by the equation $x^2 - y^2 = 4$.

Show that : a) $\dfrac{dy}{dx} = \dfrac{x}{y}$ b) $\dfrac{d^2y}{dx^2} = \dfrac{y^2 - x^2}{y^3}$

4. y is defined implicitly in terms of x by the equation $y^2 - x^2 = 2x$.

Show that : a) $\dfrac{dy}{dx} = \dfrac{1 + x}{y}$ b) $\dfrac{d^2y}{dx^2} = \dfrac{y^2 - (1 + x)^2}{y^3}$

5. y is defined implicitly in terms of x by the equation $xy = e^x$.

Show that : a) $\dfrac{dy}{dx} = \dfrac{e^x - y}{x}$ b) $\dfrac{d^2y}{dx^2} = \dfrac{(x - 2)e^x + 2y}{x^2}$

6. y is defined implicitly in terms of x by the equation $xy + y^2 = 1$.

Show that : a) $\dfrac{dy}{dx} = -\dfrac{y}{(x + 2y)}$ b) $\dfrac{d^2y}{dx^2} = \dfrac{2y(x + y)}{(x + 2y)^3}$

Logarithmic Differentiation

$$\ln(ab) = \ln a + \ln b$$
$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$
$$\ln\left(a^n\right) = n\ln a$$

If a function involves the product or quotient of powers or roots

- take the natural log (ln) of the function before differentiating.

SIMPLIFY THE FUNCTION BEFORE DIFFERENTIATING.

Examples – use logarithmic differentiation on the following to find $\frac{dy}{dx}$.

1) $y = 10^x$

$$\ln y = \ln\left(10^x\right)$$
$$\ln y = x\ln 10$$
$$\frac{1}{y}\frac{dy}{dx} = \ln 10$$
$$\frac{dy}{dx} = y\ln 10$$
$$y = 10^x$$
$$\frac{dy}{dx} = 10^x \ln 10$$

2) $y = 2^{3x}$

$$\ln y = \ln\left(2^{3x}\right)$$
$$\ln y = 3x\ln 2$$
$$\frac{1}{y}\frac{dy}{dx} = 3\ln 2$$
$$\frac{dy}{dx} = 3y\ln 2$$
$$y = 2^{3x}$$
$$\frac{dy}{dx} = 2^{3x}3\ln 2$$

3) $y = \dfrac{x^2(x+1)^4}{\sqrt{4x+1}}$

$$\ln y = \ln\left(\frac{x^2(x+1)^4}{\sqrt{4x+1}}\right)$$
$$\ln y = \ln x^2 + \ln(x+1)^4 - \ln(4x+1)^{1/2}$$
$$\ln y = 2\ln x + 4\ln(x+1) - \frac{1}{2}\ln(4x+1)$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{4}{x+1} - \frac{4}{2(4x+1)}$$
$$\frac{dy}{dx} = y\left(\frac{2}{x} + \frac{4}{x+1} - \frac{2}{4x+1}\right)$$
$$y = \frac{x^2(x+1)^4}{\sqrt{4x+1}}$$
$$\frac{dy}{dx} = \frac{x^2(x+1)^4}{\sqrt{4x+1}}\left(\frac{2}{x} + \frac{4}{x+1} - \frac{2}{4x+1}\right)$$

4) $y = \dfrac{xe^{x^2}}{\sqrt{\sin x}}$

$$\ln y = \ln\left(\frac{xe^{x^2}}{\sqrt{\sin x}}\right)$$
$$\ln y = \ln x + \ln e^{x^2} - \ln(\sin x)^{1/2}$$
$$\ln y = \ln x + x^2 - \frac{1}{2}\ln(\sin x)$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + 2x - \frac{\cos x}{2\sin x}$$
$$\frac{dy}{dx} = y\left(\frac{1}{x} + 2x - \frac{1}{2}\cot x\right)$$
$$y = \frac{xe^{x^2}}{\sqrt{\sin x}}$$
$$\frac{dy}{dx} = \frac{xe^{x^2}}{\sqrt{\sin x}}\left(\frac{1}{x} + 2x - \frac{1}{2}\cot x\right)$$

Worksheet: Logarithmic Differentiation 1 & 2 **Scholar p19 Q55 - 62**

1. Use logarithmic differentiation to find an expression for $\frac{dy}{dx}$ in each case.

a) $y = 3^x$ b) $y = 4^{2x}$ c) $y = 10^{3x}$ d) $y = x^2(x+1)^4$

e) $y = \frac{x^4}{(x+2)^6}$ f) $y = x^2\sqrt{2x+1}$ g) $y = x^6(2x-1)^3$ h) $y = \frac{2x+1}{\sqrt{2x+3}}$

i) $y = \frac{\sqrt{4x+1}}{x^2}$ j) $y = \frac{(2x+3)^2}{\sqrt{x+1}}$ k) $y = \pi^{4x}$ l) $y = \frac{x+1}{\sqrt{2x+1}}$

m) $y = \frac{x^3}{\sqrt{x+4}}$ n) $y = \frac{x^4(x+1)^3}{(x+2)^6}$ o) $y = \frac{x\sqrt{2x+1}}{x+1}$

2. Given $y = \frac{2^x}{2x+1}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{2^x}{2x+1}\left(\ln 2 - \frac{2}{2x+1}\right)$.

3. Given $y = (2x+1)^4(2x+3)^6$, use logarithmic differentiation to show that
$$\frac{dy}{dx} = 4(2x+1)^4(2x+3)^6\left(\frac{2}{2x+1} + \frac{3}{2x+3}\right).$$

4. Given $y = \frac{5^x}{x}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{5^x}{x}\left(\ln 5 - \frac{1}{x}\right)$.

5. Given $y = \sqrt{\frac{x+1}{x-1}}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{x+1}{x-1}}\left(\frac{1}{x+1} - \frac{1}{x-1}\right)$.

6. Given $y = \frac{x^2(x^2+1)}{x^2+2}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{2x^2(x^2+1)}{x^2+2}\left(\frac{1}{x} + \frac{x}{x^2+1} - \frac{x}{x^2+2}\right)$

7. Given $y = \frac{x\sqrt{x+3}}{x+2}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{x\sqrt{x+3}}{x+2}\left(\frac{1}{x} + \frac{1}{2(x+3)} - \frac{1}{x+2}\right)$.

8. Given $y = \frac{xe^{x^2}}{x+1}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{xe^{x^2}}{x+1}\left(\frac{1}{x} + 2x - \frac{1}{x+1}\right)$.

9. Given $y = \frac{\sqrt{x}(1-x)}{(3x+1)^{\frac{2}{3}}}$, use logarithmic differentiation to show that $\frac{dy}{dx} = \frac{\sqrt{x}(1-x)}{(3x+1)^{\frac{2}{3}}}\left(\frac{1}{2x} - \frac{1}{1-x} - \frac{2}{3x+1}\right)$.

10. Given $y = x^{\sin x}$, use logarithmic differentiation to show that $\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \ln x \cos x\right)$.

ANSWERS

1.

a) $\ln 3.3^x$

b) $2\ln 4.4^{2x}$

c) $3\ln 10.10^{3x}$

d) $2x^2(x+1)^4\left(\frac{1}{x}+\frac{2}{x+1}\right)$

e) $\frac{2x^4}{(x+2)^6}\left(\frac{2}{x}-\frac{3}{x+2}\right)$

f) $x^2\sqrt{2x+1}\left(\frac{2}{x}+\frac{1}{2x+1}\right)$

g) $6x^6(2x-1)^3\left(\frac{1}{x}+\frac{1}{2x-1}\right)$

h) $\frac{2x+1}{\sqrt{2x+3}}\left(\frac{2}{2x+1}-\frac{1}{2x+3}\right)$

i) $\frac{2\sqrt{4x+1}}{x^2}\left(\frac{1}{4x+1}-\frac{1}{x}\right)$

j) $\frac{(2x+3)^2}{\sqrt{x+1}}\left(\frac{4}{2x+3}-\frac{1}{2(x+1)}\right)$

k) $4\ln\pi.\pi^{4x}$

l) $\frac{x+1}{\sqrt{2x+1}}\left(\frac{1}{x+1}-\frac{1}{2x+1}\right)$

m) $\frac{x^3}{\sqrt{x+4}}\left(\frac{3}{x}-\frac{1}{2(x+4)}\right)$

n) $\frac{x^4(x+1)^3}{(x+2)^6}\left(\frac{4}{x}+\frac{3}{x+1}-\frac{6}{x+2}\right)$

o) $\frac{x\sqrt{2x+1}}{x+1}\left(\frac{1}{x}+\frac{1}{2x+1}-\frac{1}{x+1}\right)$

1. Given $y=\frac{(2x+1)^2}{(2x+5)^3}$, use logarithmic differentiation to show that

$$\frac{dy}{dx}=\frac{2(2x+1)^2}{(2x+5)^3}\left(\frac{2}{2x+1}-\frac{3}{2x+5}\right).$$

2. Given $y=xe^{-x}\cos x$, use logarithmic differentiation to show that

$$\frac{dy}{dx}=xe^{-x}\cos x\left(\frac{1}{x}-1-\tan x\right).$$

3. Given $y=(\sin x)^x$, use logarithmic differentiation to show that

$$\frac{dy}{dx}=(\sin x)^x\left(x\cot x+\ln(\sin x)\right).$$

4. Given $y=\left(x^2+1\right)^x$, use logarithmic differentiation to show that

$$\frac{dy}{dx}=\left(x^2+1\right)^x\left(\frac{2x^2}{x^2+1}+\ln(x^2+1)\right).$$

5. Given $y=\frac{e^x\sin x}{x}$, use logarithmic differentiation to show that

$$\frac{dy}{dx}=\frac{e^x\sin x}{x}\left(1+\cot x-\frac{1}{x}\right).$$

6. Given $y=x^{x^2}$, use logarithmic differentiation to show that

$$\frac{dy}{dx}=x^{x^2+1}(1+2\ln x).$$

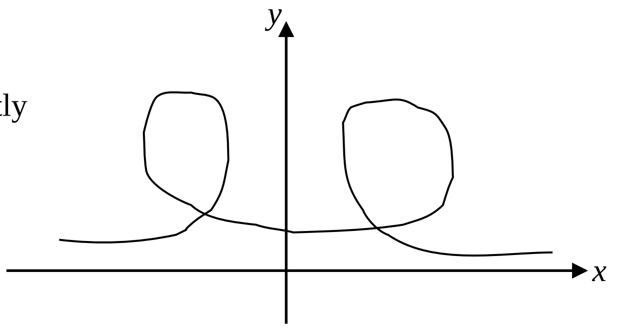
7. Given $y=\frac{\sqrt{\sin x}}{x}$, use logarithmic differentiation to show that

$$\frac{dy}{dx}=\frac{\sqrt{\sin x}}{x}\left(\frac{1}{2}\cot x-\frac{1}{x}\right).$$

PARAMETRIC EQUATIONS

Sometimes a curve can cross itself or double back.

This curve cannot be described by expressing *y* directly in terms of *x*.



A **parametric equation** is where the *x* and *y* coordinates are both written in terms of another letter. This is called a parameter and is usually given the letter *t* or θ.
(θ is normally used when the parameter is an angle, and is measured from the positive *x*-axis.)

If (x, y) is the coordinate of the point at time *t*, both *x* and *y* are functions of *t*.

$x = f(t)$ $\qquad$ $y = g(t)$ $\qquad$ ***t*** is called the parameter.

eg. $\qquad x = t^2 \qquad y = 2t$

when $t = 3 \qquad x = 9 \qquad y = 6 \qquad$ When $t = 3$, the coordinate is $(9, 6)$.

To find the gradient of the curve:

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}} \qquad \left[\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}\right]$$

$$x = t^2 \qquad y = 2t$$

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 2 \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$$

Steps for Parametric Equations

1) Take the two equations for $x =$ and $y =$ and find the derivative of each in terms of *t*.

2) To find the gradient use $\frac{dy}{dt}$ divided by the **inverse** of $\frac{dx}{dt}$.

Parametric Equations - Examples

1. A curve is defined by the parametric equations $x = t^2 + \frac{1}{t^2}$ $\quad y = t^2 - \frac{1}{t^2}$ $\quad (t \neq 0)$.

   Find an expression for $\frac{dy}{dx}$ in terms of t. Give your answer in its simplest form.

Solution

$$x = t^2 + t^{-2} \qquad y = t^2 - t^{-2}$$

$$\frac{dx}{dt} = 2t - \frac{2}{t^3} \qquad \frac{dy}{dt} = 2t + \frac{2}{t^3}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{2t + \frac{2}{t^3}}{2t - \frac{2}{t^3}} \qquad (\times \text{ by } t^3)$$

$$\frac{dy}{dx} = \frac{2t^4 + 2}{2t^4 - 2} \qquad (\div \text{ by } 2)$$

$$\underline{\underline{\frac{dy}{dx} = \frac{t^4 + 1}{t^4 - 1}}}$$

2. A curve is defined by the parametric equations $x = \theta - \sin\theta$ $\quad y = 1 - \cos\theta$ $\quad (0 \leq \theta \leq 2\pi)$.

   Find an expression for $\frac{dy}{dx}$ in terms of θ.

Solution

$$x = \theta - \sin\theta \qquad y = 1 - \cos\theta$$

$$\frac{dx}{d\theta} = 1 - \cos\theta \qquad \frac{dy}{d\theta} = \sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$\underline{\underline{\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta}}}$$

Parametric Equations - Examples

3. A curve is defined by the parametric equations $x = \dfrac{t}{1+t}$ $y = \dfrac{1+t}{1-t}$ $(t \neq \pm 1)$.

   Find an expression for $\dfrac{dy}{dx}$ in terms of t. Give your answer in its simplest form.

Solution

$$x = \frac{t}{1+t} \qquad u = t \quad u' = 1 \qquad v = 1+t \quad v' = 1$$

$$\frac{dx}{dt} = \frac{u'v - uv'}{v^2}$$

$$\frac{dx}{dt} = \frac{1+t-t}{(1+t)^2}$$

$$\frac{dx}{dt} = \frac{1}{(1+t)^2}$$

$$y = \frac{1+t}{1-t} \qquad u = 1+t \quad u' = 1 \qquad v = 1-t \quad v' = 1$$

$$\frac{dy}{dt} = \frac{u'v - uv'}{v^2}$$

$$\frac{dy}{dt} = \frac{1-t+1+t}{(1-t)^2}$$

$$\frac{dy}{dt} = \frac{2}{(1-t)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{2}{(1-t)^2} \times \frac{(1+t)^2}{1}$$

$$\frac{dy}{dx} = \frac{2(1+t)^2}{(1-t)^2}$$

$$\underline{\underline{\frac{dy}{dx} = 2\left(\frac{1+t}{1-t}\right)^2}}$$

Worksheet: Parametric Differentiation 1 **Scholar p24 Q71 - 76**

1. In each case, a curve is defined by the given parametric equations. Find an expression for $\frac{dy}{dx}$ in terms of the parameter t and simplify your answer where possible.

a) $x = 5t^2, \quad y = 10t$

b) $x = t^2 + 1, \quad y = 4t^3$

c) $x = 6t^2, \quad y = t^3$

d) $x = t^2 + t, \quad y = t^3$

e) $x = t^3 + t^2, \quad y = t^2 + t$

f) $x = (t+1)^2, \quad y = t^2 - 1$

g) $x = 4t^2 + 3, \quad y = 2t^3$

h) $x = t^2, \quad y = t(t^2 - 1)$

i) $x = t - \cos t, \quad y = 1 + \sin t$

j) $x = 2\sin t, \quad y = 2\cos t$

k) $x = at^2, \quad y = 2at$

l) $x = 3t^2 + 4t - 5, \quad y = t^3 + t^2$

m) $x = \sqrt{t}, \quad y = t^2 - 3$

n) $x = t^2, \quad y = \frac{1}{t}$

o) $x = e^t, \quad y = e^{2t}$

p) $x = 2t, \quad y = \frac{2}{t}$

q) $x = \frac{1}{t}, \quad y = \sqrt{t^2 + 1}$

r) $x = 4\cos t, \quad y = 2\sin t$

s) $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}$

t) $x = e^t \cos t, \quad y = e^t \sin t$

u) $x = \cos t, \quad y = \sin^2 t$

v) $x = \frac{2}{t+1}, \quad y = \frac{4}{t-1}$

w) $x = \frac{t-1}{t+1}, \quad y = \frac{2t-1}{t-2}$

x) $x = \frac{t}{t+1}, \quad y = \frac{t^2}{t+1}$

y) $x = \frac{1}{1+t}, \quad y = \frac{t}{1-t}$

z) $x = \frac{t}{1+t^2}, \quad y = \frac{t}{1-t^2}$

2. A curve is defined by the parametric equations $x = \sin\theta + \cos\theta, \quad y = \sin\theta - \cos\theta.$

Show that $\frac{dy}{dx} = \frac{1 + \tan\theta}{1 - \tan\theta}.$

3. A curve is defined by the parametric equations $x = 4\sin\theta, \quad y = \cos 2\theta.$

Show that $\frac{dy}{dx} = -\sin\theta.$

4. A curve is defined by the parametric equations $x = \cos^2\theta, \quad y = \sin^3\theta.$

Show that $\frac{dy}{dx} = -\frac{3}{2}\sin\theta.$

5. A curve is defined by the parametric equations $x = \sec\theta, \quad y = \tan\theta.$

Show that $\frac{dy}{dx} = \operatorname{cosec}\theta.$

ANSWERS

1.a) $\frac{1}{t}$ b) $6t$ c) $\frac{t}{4}$

d) $\frac{3t^2}{2t+1}$ e) $\frac{2t+1}{t(3t+2)}$ f) $\frac{t}{t+1}$

g) $\frac{3t}{4}$ h) $\frac{3t^2-1}{2t}$ i) $\frac{\cos t}{1+\sin t}$

j) $-\tan t$ k) $\frac{1}{t}$ l) $\frac{t}{2}$

m) $4t^{3/2}$ n) $-\frac{1}{2t^3}$ o) $2e^t$

p) $-\frac{1}{t^2}$ q) $-\frac{t^3}{\sqrt{t^2+1}}$ r) $-\frac{1}{2}\cot t$

s) $\frac{t^2+1}{t^2-1}$ t) $\frac{\cos t+\sin t}{\cos t-\sin t}$ u) $-2\cos t$

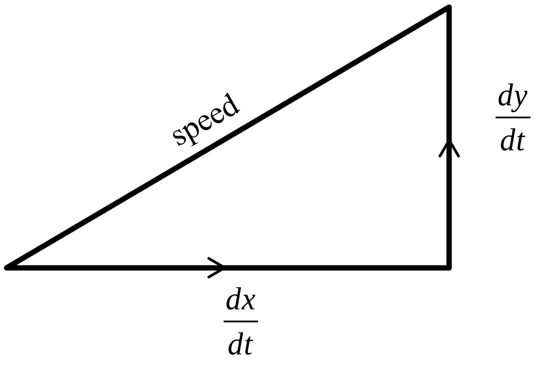
v) $2\left(\frac{t+1}{t-1}\right)^2$ w) $-\frac{3}{2}\left(\frac{t+1}{t-2}\right)^2$ x) $t(t+2)$

y) $-\left(\frac{1+t}{1-t}\right)^2$ z) $\left(\frac{1+t^2}{1-t^2}\right)^3$

MOTION

If the position of a particle is given as parametric equations $x = f(t)$ $y = g(t)$
velocity can be split into

Horizontal Velocity $\dfrac{dx}{dt}$ Vertical Velocity $\dfrac{dy}{dt}$



$$\boxed{\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}$$

Scholar p25 Q77, 78

Equations of Tangents

Equations of Tangents can also be deduced using Parametric Equations.

- Calculate the derivative in order to find the gradient.
- Using the defined parameter, substitute into the parametric equations to find the coordinate (x, y)
- Substitute into the equation of a line.

1) A curve is defined by the parametric equation $x = t^2 + 1,\quad y = t\left(t^2 + 1\right)$ for all t.

Find the equation of the tangent to the curve at the point with parameter $t = 2$.

$x = t^2 + 1 \qquad y = t^3 + t$

$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 3t^2 + 1$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 + 1}{2t}$$

$t = 2$

$\frac{dy}{dx} = \frac{12 + 1}{4} = \frac{13}{4}$

$x = 4 + 1 = 5$

$y = 8 + 2 = 10$

$y - b = m(x - a)$

$y - 10 = \frac{13}{4}(x - 5)$

$4y - 40 = 13x - 65$

$\underline{\underline{4y = 13x - 25}}$

2) A curve is defined by the parametric equation $x = t^2 + t - 1,\quad y = 2t^2 - t + 2$ for all t.

Show that the point $A(-1, 5)$ lies on the curve and find the equation of the tangent to the curve at the point A.

$x = t^2 + t + 1 \qquad y = 2t^2 - t + 2$

$\frac{dx}{dt} = 2t + 1 \qquad \frac{dy}{dt} = 4t - 1$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t - 1}{2t + 1}$$

$x = -1$

$t^2 + t - 1 = 0$

$t^2 + t = 0$

$t(t + 1) = 0$

$\underline{\underline{t = 0 \quad t = -1}}$

$y = 5$

$2t^2 - t + 2 = 5$

$2t^2 - t - 3 = 0$

$(2t - 3)(t + 1) = 0$

$\underline{\underline{t = \frac{3}{2} \quad t = -1}}$

$t = -1$

$\frac{dy}{dx} = \frac{-4 - 1}{-2 + 1} = \frac{-5}{-1} = 5$

When $t = -1$ $(-1, 5)$

$y - b = m(x - a)$

$y - 5 = 5(x + 1)$

$y - 5 = 5x + 5$

$\underline{\underline{y = 5x + 10}}$

Worksheet: Parametric Differentiation 2

Scholar p27 Q79 - 83

1. A curve is defined by the parametric equations $x = t^2, \quad y = 2t.$

   Find the equation of the tangent to the curve at the point with parameter $t = 3$.

2. A curve is defined by the parametric equations $x = t^2 + 1, \quad y = 2t^3.$

   Find the equation of the tangent to the curve at the point with parameter $t = 1$.

3. A curve is defined by the parametric equations $x = t^2 + 2t - 1, \quad y = t^2 - 4t + 1.$

   Find the equation of the tangent to the curve at the point with parameter $t = 0$.

4. A curve is defined by the parametric equations $x = t^2 - 3, \quad y = 2t^3.$

   Find the equation of the tangent to the curve at the point with parameter $t = 1$.

5. A curve is defined by the parametric equations $x = t^2, \quad y = t\left(t^2 + 1\right).$

   Find the equation of the tangent to the curve at the point with parameter $t = 1$.

6. A curve is defined by the parametric equations $x = 2t^2 + t - 5, \quad y = t^2 + 3t + 1.$

   Find the equation of the tangent to the curve at the point with parameter $t = 1$.

7. A curve is defined by the parametric equations $x = t(t + 1), \quad y = t^2(t - 1).$

   Find the equation of the tangent to the curve at the point with parameter $t = 1$.

8. In each case, a curve is defined by the given parametric equations. Find the coordinates of the stationary point(s) on each curve. (You do not need to determine the nature of the stationary points.)

   a) $x = t^2 + t - 1, \quad y = t^2 - 4t + 1$

   b) $x = t^2 + 1, \quad y = 2t(t - 2)$

   c) $x = t^3 - 5t, \quad y = t^2 - 6t + 2$

   d) $x = t^2 + t, \quad y = t^3 - 12t$

   e) $x = t^2 + \dfrac{2}{t}, \quad y = t^2 - \dfrac{2}{t}$

   f) $x = t - \dfrac{1}{t}, \quad y = t + \dfrac{1}{t}$

   g) $x = \left(t^2 + 1\right)^2, \quad y = t(t - 2)$

   h) $x = t^2 + 1, \quad y = t(t - 3)^2$

ANSWERS

1. $3y = x + 9$

2. $y = 3x - 4$

3. $y = -2x - 1$

4. $y = 3x + 8$

5. $y = 2x$

6. $y = x + 7$

7. $3y = x - 2$

8.a) $(5,-3)$ b) $(2,-2)$

c) $(12,-7)$ d) $(6,-16)$ and $(2,16)$

e) $(-1,3)$ f) $(0,2)$ and $(0,-2)$

g) $(4,-1)$ h) $(2,4)$ and $(10,0)$

SECOND DERIVATIVES

Consider $x = t^2$ $\quad y = 2t$

$\frac{dx}{dt} = 2t \quad \frac{dy}{dt} = 2$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{2t} = \frac{1}{t}.$$

To find the second derivative

$$\frac{dy}{dx} = \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{1}{t}\right).\frac{dt}{dx} \qquad \frac{dt}{dx} = \frac{1}{dx/dt} = \frac{1}{2t}$$

$$\frac{d^2y}{dx^2} = -t^{-2}\left(\frac{1}{2t}\right)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{t^2}\left(\frac{1}{2t}\right) = -\frac{1}{2t^3}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx}}$$

Example: A curve is defined by the parametric equations $x = \frac{2t}{1-t^2}$, $\quad y = \frac{1+t^2}{1-t^2} \quad (t \neq \pm 1)$.

Show that $\frac{dy}{dx} = \frac{2t}{1+t^2}$ and $\frac{d^2y}{dx^2} = \left[\frac{1-t^2}{1+t^2}\right]^3$.

Worksheet: Parametric Differentiation 3 **Scholar p28 Q84 - 88**

1. In each case, a curve is defined by the given parametric equations. Find an expression for the second derivative $\dfrac{d^2y}{dx^2}$ in terms of the parameter t.

a) $x = t^2, \quad y = 2t$

b) $x = t^2, \quad y = \ln t$

c) $x = \dfrac{2}{t}, \quad y = 2t^3 + 1$

d) $x = 2t^3 - 1, \quad y = 3t^2$

e) $x = t^2, \quad y = \dfrac{4}{t}$

f) $x = t^2 + t, \quad y = t^2 - t$

2. A curve is defined by the parametric equations $x = t + \sin t, \quad y = t - \cos t$.

Show that $\dfrac{d^2y}{dx^2} = \dfrac{1 + \sin t + \cos t}{(1 + \cos t)^3}$.

3. A curve is defined by the parametric equations $x = t^2 + 1, \quad y = t(t^2 + 1)$.

Show that $\dfrac{d^2y}{dx^2} = \dfrac{3t^2 - 1}{4t^3}$.

ANSWERS

1.a) $-\dfrac{1}{2t^3}$

b) $-\dfrac{1}{2t^4}$

c) $6t^5$

d) $-\dfrac{1}{6t^3}$

e) $\dfrac{3}{t^5}$

f) $\dfrac{4}{(2t+1)^3}$