

Algebraic skills (1.1)

Expressing proper rational functions as a sum of partial fractions (denominator of degree at most 3 and easily factorised).

DEFINITIONS

Polynomial – an expression constructed from variables (eg x^2 , x) and constants.

Rational Function – a fraction whose numerator and denominator are polynomials. eg. $\frac{4x^3 + 2x - 5}{6x^4 - 3x^2 + 11}$

Proper Rational Function – the degree of the numerator **MUST** be less than that of the denominator.

eg. $\frac{2x+1}{3x^2-6x+10}, \frac{x^4-15}{x^6+3x^4-2x+5}$

Improper Rational Function – the degree of the numerator is **EQUAL** to or **MORE** than the denominator.

eg. $\frac{4x^2+2x-3}{x^2-9x+20}, \frac{x^4-5x^3+2x^2+14}{2x^3+x^2-3x+8}$

The operation of adding fractions such as $\frac{2}{x+1}$ and $\frac{3}{x+2}$ is a simple algebraic process:

$$\frac{2}{x+1} + \frac{3}{x+2} = \frac{2(x+2) + 3(x+1)}{(x+1)(x+2)} = \frac{5x+7}{(x+1)(x+2)}$$

The reverse process however, of splitting a fraction into two or more **PARTIAL FRACTIONS** is not as straightforward. It is sometimes necessary to decompose a rational function into the sum of simpler fractions in order to perform linear operations such as integration.

Partial fractions can be categorised into 4 types depending on the form of the denominator.

PROPER FUNCTIONS (Power of denominator $>$ than that of the numerator)

TYPE 1: Distinct Linear Factors: the denominator only includes linear factors eg $(x+2)(x-3)$

TYPE 2: Repeated Linear Factor: the denominator includes some repeated factors eg $(x+2)^2$

TYPE 3: Irreducible Quadratic Factor: the denominator includes at least one non-linear factor (cannot be reduced – use $b^2 - 4ac$). eg $x^2 + 2x + 2$.
Must show that factor is irreducible.

IMPROPER FUNCTIONS (Power of denominator $\leq$ than that of the numerator)

TYPE 4: Improper Rational Factor: Requires *algebraic long division* and then partial fraction.

eg. $\frac{2x^3 + 5x^2 - 2x - 1}{x^3 - x}$

Algebraic Long Division

If we have a rational function $\frac{p(x)}{q(x)}$ where the degree (highest power) of $q(x)$ is either the same or less than the degree of $p(x)$ then we can rewrite it in the form of $r(x) + \frac{s(x)}{d(x)}$ where the degree of $s(x)$ is less than $q(x)$.

1)

$$\frac{3x^3 - 2x^2 + 4x - 3}{x^2 + 3x + 3}$$

degree of 3 (pointing to $3x^3$)
degree of 2 (pointing to x^2)

$$3x - 11 \quad \text{Quotient}$$

$$\begin{array}{r} x^2 + 3x + 3 \overline{) 3x^3 - 2x^2 + 4x - 3} \\ \underline{3x^3 + 9x^2 + 9x} \quad \text{Subtract} \\ -11x^2 - 5x - 3 \\ \underline{-11x^2 - 33x - 33} \quad \text{Subtract} \\ 28x + 30 \end{array}$$

Divisor (pointing to $x^2 + 3x + 3$)
Remainder (pointing to $28x + 30$)

$$\frac{3x^3 - 2x^2 + 4x - 3}{x^2 + 3x + 3} = 3x - 11 + \frac{28x + 30}{x^2 + 3x + 3}$$

The numerator now has a degree less than that of the denominator.

2)

$$\frac{x^3 - 1}{x + 2}$$

$$x + 2 \overline{) x^3 - 2x + 4}$$

$$\begin{array}{r} x^3 + 0x^2 + 0x - 1 \\ \underline{x^3 + 2x^2} \\ -2x^2 + 0x - 1 \\ \underline{-2x^2 - 4x} \\ 4x - 1 \\ \underline{4x + 8} \\ -9 \end{array}$$

$$\frac{x^3 - 1}{x + 2} = x^2 - 2x + 4 - \frac{9}{x + 2}$$

1.
$$\frac{x^2 + 3x + 5}{x + 2}$$

2.
$$\frac{x^2 - 2x + 4}{x + 3}$$

3.
$$\frac{x^2 + 3x - 5}{x - 2}$$

4.
$$\frac{x^2 - 2x - 1}{x + 1}$$

5.
$$\frac{x^2 + x - 3}{x - 1}$$

6.
$$\frac{x^2 + 2x - 7}{x + 4}$$

7.
$$\frac{2x^2 + 7x + 2}{x + 3}$$

8.
$$\frac{2x^2 - x + 3}{x + 1}$$

9.
$$\frac{2x^2 - 5x + 7}{x - 1}$$

10.
$$\frac{3x^2 - 5x + 1}{x - 4}$$

11.
$$\frac{2x^2 + 5x + 1}{x + 2}$$

12.
$$\frac{x^2 + x + 3}{x^2 + x - 1}$$

13.
$$\frac{3x^2 + 4x + 5}{x^2 + x + 1}$$

14.
$$\frac{x^2 - x + 1}{x^2 + x - 2}$$

15.
$$\frac{2x^2 + 3x + 1}{x^2 + x + 2}$$

16.
$$\frac{x^3 + 3x^2 + 4x - 5}{x^2 + 1}$$

17.
$$\frac{x^2}{x^2 - x + 2}$$

18.
$$\frac{x^3 + 1}{x + 2}$$

19.
$$\frac{x^2 + 3}{x^2 - 4}$$

20.
$$\frac{x^3 - 2x^2 - 6x - 11}{x - 4}$$

21.
$$\frac{x^4 + 3x^3 + 2x^2 - 3}{x^2 + 2x}$$

22.
$$\frac{x^4 - 2x + 5}{x^2 + 4}$$

23.
$$\frac{x^3 + 3x^2 + 7}{x^2 - 2x}$$

24.
$$\frac{x^3 + 4x^2 - x + 2}{x^2 + x}$$

25.
$$\frac{3x^3 + 7x - 1}{x^2 + 3}$$

ANSWERS

1. $x+1+\frac{3}{x+2}$

2. $x-5+\frac{19}{x+3}$

3. $x+5+\frac{5}{x-2}$

4. $x-3+\frac{2}{x+1}$

5. $x+2-\frac{1}{x-1}$

6. $x-2+\frac{1}{x+4}$

7. $2x+1-\frac{1}{x+3}$

8. $2x-3+\frac{6}{x+1}$

9. $2x-3+\frac{4}{x-1}$

10. $3x+7+\frac{29}{x-4}$

11. $2x+1-\frac{1}{x+2}$

12. $1+\frac{4}{x^2+x-1}$

13. $3+\frac{x+2}{x^2+x+1}$

14. $1+\frac{-2x+3}{x^2+x-2}$

15. $2+\frac{x-3}{x^2+x+2}$

16. $x+3+\frac{3x-8}{x^2+1}$

17. $1+\frac{x-2}{x^2-x+2}$

18. $x^2-2x+4-\frac{7}{x+2}$

19. $1+\frac{7}{x^2-4}$

20. $x^2+2x+2-\frac{3}{x-4}$

21. $x^2+x-\frac{3}{x^2+2x}$

22. $x^2-4+\frac{-2x+21}{x^2+4}$

23. $x+5+\frac{10x+7}{x^2-2x}$

24. $x+3+\frac{-4x+2}{x^2+x}$

25. $3x+\frac{-2x-1}{x^2+3}$

1. Find the remainder on dividing the following:

- a) $x^3 - 2x^2 - 3x + 4$ by $x - 3$
- b) $x^3 + x^2 - 7x + 5$ by $x + 2$
- c) $2x^3 - 3x^2 - 8x + 5$ by $2x + 3$
- d) $2x^3 + 3x^2 - 3x - 2$ by $2x + 1$
- e) $x^3 - 6x^2 + 11x - 6$ by $x - 3$

2. Find the factors for the following:

- a) $x^3 + 2x^2 - 5x - 6$
- b) $x^3 - 7x - 6$
- c) $2x^3 - 3x^2 - 3x + 2$
- d) $2x^3 + 9x^2 + 13x + 6$
- e) $6x^3 - 13x^2 - 14x - 3$
- f) $x^4 - x^3 - 11x^2 + 9x + 18$
- g) $4x^4 - 31x^2 + 21x + 18$

3. Solve the following equations

- a) $2x^3 - 7x^2 - 3x + 18 = 0$
- b) $4x^3 - 4x^2 - 5x + 3 = 0$
- c) $6x^3 + 17x^2 - 5x - 6 = 0$
- d) $x^4 - 15x^2 - 10x + 24 = 0$
- e) $6x^4 + 23x^3 + 23x^2 - 2x - 8 = 0$

ANSWERS

1. Remainders

- a) 4
- b) 15
- c) 3.5
- d) 0
- e) 0

2. Factors

- a) $(x+1)(x+3)(x-2)$
- b) $(x+1)(x+2)(x-3)$
- c) $(x+1)(x-2)(2x-1)$
- d) $(x+1)(x+2)(2x+3)$
- e) $(x-3)(3x+1)(2x+1)$
- f) $(x+1)(x-2)(x-3)(x+3)$
- g) $(x+3)(x-2)(2x-3)(2x+1)$

3. Solutions

- a) $x = 2, -\frac{3}{2}, 3$
- b) $x = -1, \frac{1}{2}, \frac{3}{2}$
- c) $x = -3, -\frac{1}{2}, \frac{2}{3}$
- d) $x = 1, -2, -3, 4$
- e) $x = -1, -2, -\frac{4}{3}, \frac{1}{2}$

TYPE 1: Distinct Linear Factors:

Example 1 Express $\frac{5x-8}{(x-1)(x-2)}$ in partial fractions.

$$\frac{5x-8}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

$$\frac{5x-8}{(x-1)(x-2)} = \frac{A(x-2) + B(x-1)}{(x-1)(x-2)}$$

$$5x-8 = A(x-2) + B(x-1)$$

If $x=1$ then $-3 = -A \Rightarrow A=3$

If $x=2$ then $2 = B(2-1) \Rightarrow B=2$

Ans : $\frac{5x-8}{(x-1)(x-2)} = \frac{3}{(x-1)} + \frac{2}{(x-2)}$

Split into the sum of two algebraic fractions with constants A, B etc as numerator

Add the algebraic fractions to result in the same denominator as LHS.

Ignore denominator and solve to find constants.

Find values for A and B by substituting $x=1$ and $x=2$

Substitute values into separate algebraic fractions and write answer in full.

Examples

a)) $\frac{3x-2}{(x+2)(x-2)}$

$$\frac{3x-2}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$3x-2 = A(x-2) + B(x+2)$$

$x=-2 \quad -8 = -4A \quad A=2$

$x=2 \quad 4 = 4B \quad B=1$

$$\frac{3x-2}{(x+2)(x-2)} = \frac{2}{x+2} + \frac{1}{x-2}$$

TYPE 1: Distinct Linear Factors:

b) $\frac{6x-15}{x(x+3)}$

$$\frac{6x-15}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}$$

$$6x-15 = A(x+3) + Bx$$

$$x=0 \quad -15=3A \quad A=-5$$

$$x=-3 \quad -33=-3B \quad B=11$$

$$\frac{6x-15}{x(x+3)} = \frac{11}{x+3} - \frac{5}{x}$$

c) $\frac{4x+1}{2x^2+3x+1}$

$$\frac{4x+1}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1}$$

$$4x+1 = A(x+1) + B(2x+1)$$

$$x = -\frac{1}{2} \quad -1 = \frac{1}{2}A \quad A = -2$$

$$x = -1 \quad -3 = -B \quad B = 3$$

$$\frac{4x+1}{2x^2+3x+1} = \frac{3}{x+1} - \frac{2}{2x+1}$$

d) $\frac{2x^2-2x-6}{(x+1)(x+2)(x-1)}$

$$\frac{2x^2-2x-6}{(x+1)(x+2)(x-1)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x-1}$$

$$2x^2-2x-6 = A(x+2)(x-1) + B(x+1)(x-1) + C(x+1)(x+2)$$

$$x = -1 \quad -2 = -2A \quad A = 1$$

$$x = -2 \quad 6 = 3B \quad B = 2$$

$$x = 1 \quad -6 = 6C \quad C = -1$$

$$\frac{2x^2-2x-6}{(x+1)(x+2)(x-1)} = \frac{1}{x+1} + \frac{2}{x+2} - \frac{1}{x-1}$$

Type 1 – Distinct Linear Factors

1. $\frac{2x}{(x-1)(x+1)}$

2. $\frac{10x}{(x-2)(x+3)}$

3. $\frac{4x}{(x+1)(x+5)}$

4. $\frac{20}{(x-3)(x+2)}$

5. $\frac{3}{(x-1)(x+2)}$

6. $\frac{5}{(x+2)(x+3)}$

7. $\frac{5x-14}{(x-2)(x-3)}$

8. $\frac{3x-2}{(x+2)(x-2)}$

9. $\frac{15-2x}{(2x-1)(x+3)}$

10. $\frac{7x+5}{(x-1)(x+3)}$

11. $\frac{-x-7}{(x-3)(x+1)}$

12. $\frac{6x-15}{x(x+3)}$

13. $\frac{3x+4}{x(x+2)}$

14. $\frac{-3x-12}{x(x+6)}$

15. $\frac{4-6x}{x(x+4)}$

16. $\frac{7x-11}{(5-x)(x+1)}$

17. $\frac{11x+6}{(3x+1)(2x+3)}$

18. $\frac{2x+7}{(2x-3)(4x-1)}$

19. $\frac{5x-3}{x^2+x-30}$

20. $\frac{5x-11}{x^2-4x+3}$

21. $\frac{4x+5}{x^2+3x+2}$

22. $\frac{4x-10}{x^2-3x}$

23. $\frac{6x-1}{4x^2-1}$

24. $\frac{4x+1}{2x^2+3x+1}$

25. $\frac{2x^2-2x-6}{(x+1)(x+2)(x-1)}$

ANSWERS

1. $\frac{1}{x-1} + \frac{1}{x+1}$

2. $\frac{4}{x-2} + \frac{6}{x+3}$

3. $\frac{-1}{x+1} + \frac{5}{x+5}$

4. $\frac{4}{x-3} - \frac{4}{x+2}$

5. $\frac{1}{x-1} - \frac{1}{x+2}$

6. $\frac{5}{x+2} - \frac{5}{x+3}$

7. $\frac{4}{x-2} + \frac{1}{x-3}$

8. $\frac{2}{x+2} + \frac{1}{x-2}$

9. $\frac{4}{2x-1} - \frac{3}{x+3}$

10. $\frac{3}{x-1} + \frac{4}{x+3}$

11. $\frac{-5}{2(x-3)} + \frac{3}{2(x+1)}$

12. $\frac{11}{x+3} - \frac{5}{x}$

13. $\frac{2}{x} + \frac{1}{x+2}$

14. $-\frac{2}{x} - \frac{1}{x+6}$

15. $\frac{1}{x} - \frac{7}{x+4}$

16. $\frac{4}{5-x} - \frac{3}{x+1}$

17. $\frac{1}{3x-1} + \frac{3}{2x+3}$

18. $\frac{2}{2x-3} - \frac{3}{4x-1}$

19. $\frac{2}{x-5} + \frac{3}{x+6}$

20. $\frac{3}{x-1} + \frac{2}{x-3}$

21. $\frac{1}{x+1} + \frac{3}{x+2}$

22. $\frac{10}{3x} + \frac{2}{3(x-3)}$

23. $\frac{1}{2x-1} + \frac{2}{2x+1}$

24. $\frac{3}{x+1} - \frac{2}{2x+1}$

25. $\frac{1}{x+1} + \frac{2}{x+2} - \frac{1}{x-1}$

TYPE 2: Repeated Linear Factor:

Example 1 Express $\frac{4x+1}{(x+1)(2x+1)^2}$ in partial fractions.

$$\frac{4x+1}{(x+1)(2x+1)^2} = \frac{A}{(x+1)} + \frac{B}{(2x+1)} + \frac{C}{(2x+1)^2}$$

The repeated factor is repeated as denominators increasing from the power 1 until n.

$$\frac{4x+1}{(x+1)(2x+1)^2} = \frac{A(2x+1)^2 + B(x+1)(2x+1) + C(x+1)}{(x+1)(2x+1)^2}$$

Add the algebraic fractions to result in the same denominator as LHS.

$$4x+1 = A(2x+1)^2 + B(x+1)(2x+1) + C(x+1)$$

Ignore denominator and solve to find constants.

$$\text{If } x = -1 \text{ then } -3 = A(-2+1)^2 \Rightarrow A = -3$$

$$\text{If } x = -\frac{1}{2} \text{ then } -1 = C(-\frac{1}{2}+1) \Rightarrow C = -2$$

Substitute for x to find A, B and C.

$$\text{If } x = 1 \text{ then } 5 = 9A + 6B + 2C$$

$$5 = -27 - 4 + 6B \Rightarrow B = 6$$

$$\frac{4x+1}{(x+1)(2x+1)^2} = \frac{6}{2x+1} - \frac{2}{(2x+1)^2} - \frac{3}{x+1}$$

Substitute values into separate algebraic fractions and write answer in full.

EXAMPLES

a) $\frac{3x^2 - 11x + 5}{(x-2)(x-1)^2}$

b) $\frac{x-2}{x^2(3x-2)}$

$$\frac{3x^2 - 11x + 5}{(x-2)(x-1)^2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$3x^2 - 11x + 5 = A(x-1)^2 + B(x-2)(x-1) + C(x-2)$$

$$x = 2 \quad -5 = A \quad A = -5$$

$$x = 1 \quad -3 = -C \quad C = 3$$

$$x = 0 \quad 5 = -5 + 2B - 6 \quad B = 8$$

$$\frac{3x^2 - 11x + 5}{(x-2)(x-1)^2} = \frac{8}{x-1} + \frac{3}{(x-1)^2} - \frac{5}{x-2}$$

$$\frac{x-2}{x^2(3x-2)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{3x-2}$$

$$x-2 = A(3x-2) + Bx(3x-2) + Cx^2$$

$$x = 0 \quad -2 = -2A \quad A = 1$$

$$x = -1 \quad -3 = -5 + 5B + C$$

$$x = 2 \quad 0 = 4 + 8B + 4C$$

$$\frac{x-2}{x^2(3x-2)} = \frac{1}{x^2} + \frac{1}{x} - \frac{3}{3x-2}$$

$$5B + C = 2$$

$$8B + 4C = -4$$

$$20B + 4C = 8$$

$$8B + 4C = -4$$

$$12B = 12$$

$$B = 1$$

$$C = -3$$

Type 2 – Repeated Linear Factor

1.
$$\frac{x^2 - 9x - 2}{(x+1)(x-1)^2}$$

2.
$$\frac{3x^2 - 11x + 5}{(x-2)(x-1)^2}$$

3.
$$\frac{4x^2 + 9}{(x-2)(x+3)^2}$$

4.
$$\frac{2x - 7}{(x+1)(x-2)^2}$$

5.
$$\frac{6x^2 + x - 7}{(x-3)(x+2)^2}$$

6.
$$\frac{x^2 + 6x - 3}{x(x-1)^2}$$

7.
$$\frac{16}{(x+1)(x-3)^2}$$

8.
$$\frac{5x+1}{x(2x+1)^2}$$

9.
$$\frac{2x+5}{(1-x)(x+2)^2}$$

10.
$$\frac{4x^2 - 8x + 15}{(x+1)(x-2)^2}$$

11.
$$\frac{2x^2 + 7x + 3}{x(x+1)^2}$$

12.
$$\frac{2}{(x+1)(x-1)^2}$$

13.
$$\frac{3x^2 - x - 1}{(x-2)(x+1)^2}$$

14.
$$\frac{7x^2 + 1}{x^2(x-1)}$$

ANSWERS

1. $\frac{2}{x+1} - \frac{1}{x-1} - \frac{5}{(x-1)^2}$

2. $-\frac{5}{x-2} + \frac{8}{x-1} + \frac{3}{(x-1)^2}$

3. $\frac{1}{x-2} + \frac{3}{x+3} - \frac{9}{(x+3)^2}$

4. $-\frac{1}{x+1} + \frac{1}{x-2} - \frac{1}{(x-2)^2}$

5. $\frac{2}{x-3} + \frac{4}{x+2} - \frac{3}{(x+2)^2}$

6. $-\frac{3}{x} + \frac{4}{x-1} - \frac{4}{(x-1)^2}$

7. $\frac{1}{x+1} - \frac{1}{x-3} + \frac{4}{(x-3)^2}$

8. $\frac{1}{x} - \frac{2}{2x+1} + \frac{3}{(2x+1)^2}$

9. $\frac{7}{9(1-x)} + \frac{7}{9(x+2)} + \frac{1}{3(x+2)^2}$

10. $\frac{3}{x+1} + \frac{1}{x-2} + \frac{5}{(x-2)^2}$

11. $\frac{3}{x} - \frac{1}{x+1} + \frac{2}{(x+1)^2}$

12. $\frac{1}{2(x+1)} - \frac{1}{2(x-1)} + \frac{1}{(x-1)^2}$

13. $\frac{1}{x-2} + \frac{2}{x+1} - \frac{1}{(x+1)^2}$

14. $\frac{8}{x-1} - \frac{1}{x} - \frac{1}{x^2}$

TYPE 3: Irreducible Quadratic Factor

Example 1 Express $\frac{7x+5}{(x-3)(x^2+4)}$ in partial fractions.

(x^2+4) is irreducible as it cannot be factorised into linear factors. First step is to show that it is irreducible using the DISCRIMINANT.

$$\boxed{b^2 - 4ac} \quad a = 1, b = 0 \text{ and } c = 4 \quad 0 - (4 \times 1 \times 4) = -16 \quad b^2 - 4ac < 0 \text{ therefore irreducible.}$$

$$\frac{7x+5}{(x-3)(x^2+4)} = \frac{A}{(x-3)} + \frac{Bx+C}{(x^2+4)}$$

Note the numerator for the irreducible factor is $Bx + C$.

$$\frac{7x+5}{(x-3)(x^2+4)} = \frac{A(x^2+4) + (Bx+C)(x-3)}{(x-3)(x^2+4)}$$

$$7x+5 = A(x^2+4) + (Bx+C)(x-3)$$

If $x=3$ then $26=13A \Rightarrow A=2$

If $x=0$ then $5=4(A)+(-3C)$

$$5=8-3C \Rightarrow C=1$$

If $x=1$ then $12=5(A)+(-2C)+(-2B)$

$$\text{Ans : } \underline{\underline{\frac{7x+5}{(x-3)(x^2+4)} = \frac{2}{(x-3)} + \frac{-2x+1}{(x^2+4)}}}$$

TYPE 3: Irreducible Quadratic FactorEXAMPLES

$$\text{a) } \frac{16x^2}{(x-3)(2x^2-x+1)}$$

$$\text{b) } \frac{x^2-10x-8}{x^3-8}$$

$$\frac{16x^2}{(x-3)(2x^2-x+1)} \quad a=2, b=-1, c=1$$

$$b^2 - 4ac = -7 \text{ irreducible}$$

$$\frac{16x^2}{(x-3)(2x^2-x+1)} = \frac{A}{x-3} + \frac{Bx+C}{2x^2-x+1}$$

$$16x^2 = A(2x^2-x+1) + (Bx+C)(x-3)$$

$$x=3 \quad 144 = 16A \quad A=9$$

$$x=0 \quad 0 = 9 - 3C \quad C=3$$

$$x=1 \quad 16 = 18 - 2B - 6 \quad B=-2$$

$$= \frac{9}{x-3} + \frac{3-2x}{2x^2-x+1}$$

$$\frac{x^2-10x-8}{x^3-8}$$

$$x^2+2x+4 \quad a=1, b=2, c=4$$

$$b^2 - 4ac = 4 - 16 = -12$$

Irreducible

$$x=2 \quad 8-8=0 \quad x-2 \text{ is a factor}$$

$$x-2 \quad \left| \begin{array}{r} x^2+2x+4 \\ x^3+0x^2+0x-8 \end{array} \right.$$

$$\underline{x^3-2x^2}$$

$$2x^2+0x-8$$

$$\underline{2x^2-4x}$$

$$\underline{4x-8}$$

$$\frac{x^2-10x-8}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$$

$$x^2-10x-8 = A(x^2+2x+4) + (Bx+C)(x-2)$$

$$x=2 \quad -24 = 12A \quad A=-2$$

$$x=0 \quad -8 = -8 - 2C \quad C=0$$

$$x=1 \quad -17 = -14 - B \quad B=3$$

$$= \frac{3x}{x^2+2x+4} - \frac{2}{x-2}$$

Type 3 – Irreducible Quadratic Factor

1.a) Show that the quadratic $x^2 + 2x + 2$ is irreducible.

b) Hence express $\frac{3x^2 + 2x + 1}{(x+1)(x^2 + 2x + 2)}$ in partial fractions.

2.a) Show that the quadratic $x^2 + 2x + 5$ is irreducible.

b) Hence express $\frac{8}{(1-x)(x^2 + 2x + 5)}$ in partial fractions.

3. Express in partial fractions (you must show first that the quadratic factor is irreducible).

a) $\frac{4x^2 - 3x + 2}{(x-1)(x^2 + x + 1)}$

b) $\frac{x^2 - 10x - 8}{(x-2)(x^2 + 2x + 4)}$

c) $\frac{3x - 2}{x(x^2 + 2)}$

d) $\frac{4x^2 + 5x + 13}{(x+1)(x^2 + x + 3)}$

e) $\frac{2x^2 + 3x + 1}{(x-1)(x^2 + x + 1)}$

f) $\frac{x^2 + 2x + 9}{(x-1)(x^2 + 3)}$

The following question is an actual examination question.

4.a) Factorise the cubic polynomial $x^3 - x^2 - x - 2$ as the product of a linear factor $l(x)$ and a quadratic factor $q(x)$.

b) Show that the quadratic factor $q(x)$ is irreducible.

c) Hence express $\frac{5x + 4}{x^3 - x^2 - x - 2}$ in partial fractions.

ANSWERS

1.a) Proof

b) $\frac{2}{x+1} + \frac{x-3}{x^2+2x+2}$

2.a) Proof

b) $\frac{1}{1-x} + \frac{x+3}{x^2+2x+5}$

3.a) $\frac{1}{x-1} + \frac{3x-1}{x^2+x+1}$

b) $-\frac{2}{x-2} + \frac{3x}{x^2+2x+4}$

c) $-\frac{1}{x} + \frac{x+3}{x^2+2}$

d) $\frac{4}{x+1} + \frac{1}{x^2+x+3}$

e) $\frac{2}{x-1} + \frac{1}{x^2+x+1}$

f) $\frac{3}{x-1} + \frac{-2x}{x^2+3}$

4.a) $x^3 - x^2 - x - 2 = (x-2)(x^2+x+1)$

b) Proof

c) $\frac{2}{x-2} + \frac{-2x-1}{x^2+x+1}$

Mixed Partial Fractions

1. Express in partial fractions (remember to ensure that the denominator of each fraction is factorised fully).

a) $\frac{8x-3}{x(x-1)}$

b) $\frac{3x}{(x-2)(x+1)}$

c) $\frac{4}{(x+1)(x-1)^2}$

d) $\frac{4x-9}{(x-2)(x-3)}$

e) $\frac{5x+1}{(x-1)(x^2+x+1)}$

f) $\frac{3x^2+1}{x(x+1)^2}$

g) $\frac{3x+4}{x^2+4x}$

h) $\frac{x+10}{x^2-x-12}$

i) $\frac{x^2-2x+10}{(x+2)(x-1)^2}$

j) $\frac{8}{(1-x)(x^2+2x+5)}$

k) $\frac{x+16}{2x^2+x-6}$

l) $\frac{x+1}{x^2(x-1)}$

m) $\frac{6x-1}{4x^2-1}$

n) $\frac{2x^2+7x+3}{x^3+2x^2+x}$

o) $\frac{x^2-9x-2}{(x-1)(x^2-1)}$

2. Express $\frac{9-5x}{(x-1)(x-2)(x-3)}$ in partial fractions.

3.a) Prove that x^2+2 does not factorise into the product of two linear factors with real coefficients.

b) Express $\frac{3x-2}{x(x^2+2)}$ in partial fractions.

4.a) Factorise x^3+2x^2+4x+3 into the product of a linear factor and a quadratic factor, and prove that the quadratic factor is irreducible.

b) Express $\frac{4x^2+2x+7}{x^3+2x^2+4x+3}$ in partial fractions.

5.a) Factorise x^3-3x^2+3x-2 into the product of a linear factor and a quadratic factor and prove that the quadratic factor is irreducible.

b) Express $\frac{4x^2-4x+1}{x^3-3x^2+3x-2}$ in partial fractions.

ANSWERS

1. a) $\frac{3}{x} + \frac{5}{x-1}$ b) $\frac{2}{x-2} + \frac{1}{x+1}$ c) $\frac{1}{x+1} - \frac{1}{x-1} + \frac{2}{(x-1)^2}$ d) $\frac{1}{x-2} + \frac{3}{x-3}$
- e) $\frac{2}{x-1} + \frac{-2x+1}{x^2+x+1}$ f) $\frac{1}{x} + \frac{2}{x+1} - \frac{4}{(x+1)^2}$ g) $\frac{1}{x} + \frac{2}{x+4}$ h) $\frac{2}{x-4} - \frac{1}{x+3}$
- i) $\frac{2}{x+2} - \frac{1}{x-1} + \frac{3}{(x-1)^2}$ j) $\frac{1}{1-x} + \frac{x+3}{x^2+2x+5}$ k) $\frac{5}{2x-3} - \frac{2}{x+2}$ l) $\frac{2}{x-1} - \frac{2}{x} - \frac{1}{x^2}$
- m) $\frac{1}{2x-1} + \frac{2}{2x+1}$ n) $\frac{3}{x} - \frac{1}{x+1} + \frac{2}{(x+1)^2}$ o) $\frac{2}{x+1} - \frac{1}{x-1} - \frac{5}{(x-1)^2}$

2. $\frac{2}{x-1} + \frac{1}{x-2} - \frac{3}{x-3}$

3.a) The discriminant of $x^2 + 2$ is -8 , which is less than zero, hence $x^2 + 2$ does not factorise.

b) $-\frac{1}{x} + \frac{x+3}{x^2+2}$

4.a) $x^3 + 2x^2 + 4x + 3 = (x+1)(x^2 + x + 3)$

The discriminant of $x^2 + x + 3$ is -11 , which is less than zero, hence $x^2 + x + 3$ does not factorise.

b) $\frac{3}{x+1} + \frac{x-2}{x^2+x+3}$

5.a) $x^3 - 3x^2 + 3x - 2 = (x-2)(x^2 - x + 1)$

The discriminant of $x^2 - x + 1$ is -3 , which is less than zero, hence $x^2 - x + 1$ does not factorise.

b) $\frac{3}{x-2} + \frac{x+1}{x^2-x+1}$

TYPE 4: Improper Rational Factor

Example 1 Express $\frac{2x^2 + x + 1}{(x-1)(x+1)}$ in partial fractions.

In this example notice that the degree of the numerator is NOT less than the degree of the denominator. First step must be to divide.

$$\frac{2x^2 + x + 1}{(x-1)(x+1)} = \frac{2x^2 + x + 1}{(x^2 - 1)}$$

Algebraic

Long

Division

$$x^2 - 1 \overline{) \begin{array}{r} 2x^2 + x + 1 \\ 2x^2 \\ \hline x + 3 \end{array}}$$

$$= 2 + \frac{x+3}{x^2-1}$$

$$\frac{2x^2 + x + 1}{(x^2 - 1)} = 2 + \frac{x+3}{(x-1)(x+1)}$$

Now express as partial fraction.

$$2 + \frac{x+3}{(x-1)(x+1)} = 2 + \frac{A}{(x-1)} + \frac{B}{(x+1)}$$

$$\frac{x+3}{(x-1)(x+1)} = \frac{A(x+1) + B(x-1)}{(x-1)(x+1)}$$

$$x+3 = A(x+1) + B(x-1)$$

If $x=1$ then $4=2A \Rightarrow A=2$

If $x=-1$ then $2=-2B \Rightarrow B=-1$

NOTE: Remember to include constant in answer

$$\text{Ans : } \frac{2x^2 + x + 1}{(x-1)(x+1)} = 2 + \frac{x+3}{(x-1)(x+1)} = 2 + \frac{2}{(x-1)} - \frac{1}{(x+1)}$$

TYPE 4: Improper Rational FactorEXAMPLES

$$a) \frac{x^3 - x^2 - 5x - 7}{x^2 - 2x - 3}$$

$$\frac{x^3 - x^2 - 5x - 7}{x^2 - 2x - 3}$$

$$= x + 1 - \frac{4}{x^2 - 2x - 3}$$

$$= x + 1 - \frac{4}{(x-3)(x+1)}$$

$$\frac{4}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$x^2 - 10x - 8 = A(x+1) + B(x-3)$$

$$x = 3 \quad 4 = 4A \quad A = 1$$

$$x = -1 \quad 4 = -4B \quad B = -1$$

$$= x + 1 - \left(\frac{1}{x-3} - \frac{1}{x+1} \right)$$

$$= x + 1 - \frac{1}{x-3} + \frac{1}{x+1}$$

$$x^2 - 2x - 3 \quad \left| \begin{array}{l} \frac{x+1}{x^3 - x^2 - 5x - 7} \end{array} \right.$$

$$\frac{x^3 - 2x^2 - 3x}{x^2 - 2x - 7}$$

$$\frac{x^2 - 2x - 3}{4}$$

$$\frac{x^2 - 2x - 3}{4}$$

$$b) \frac{(x+3)(x-1)}{(x+2)(x+1)}$$

$$\frac{(x+3)(x-1)}{(x+2)(x+1)}$$

$$= 1 + \frac{-x-5}{x^2 + 3x + 2}$$

$$= x + 1 - \frac{x+5}{(x+2)(x+1)}$$

$$x^2 + 3x + 2 \quad \left| \begin{array}{l} \frac{1}{x^2 + 2x - 3} \end{array} \right.$$

$$\frac{x^2 + 3x + 2}{-x-5}$$

$$-x-5$$

$$\frac{x+5}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$$

$$x + 5 = A(x+1) + B(x+2)$$

$$x = -1 \quad 4 = B \quad B = 4$$

$$x = -2 \quad 3 = -A \quad A = -3$$

$$= 1 - \left(\frac{-3}{x+2} + \frac{4}{x+1} \right)$$

$$= 1 + \frac{3}{x+2} - \frac{4}{x+1}$$

Partial Fractions for Improper Rational Functions

Using algebraic long division first, express each of the improper rational functions below as the sum of a polynomial function and partial fractions.

a)
$$\frac{x^2 - x + 6}{x^2 + x - 2}$$

b)
$$\frac{x^3 - x^2 - 5x + 1}{x^2 - 2x - 3}$$

c)
$$\frac{x^3 - 5x^2 + 11x - 12}{x^2 - 5x + 6}$$

d)
$$\frac{x^2}{x^2 - 4}$$

e)
$$\frac{x^3 + 4x^2 - x + 2}{x^2 + x}$$

f)
$$\frac{x^2 + 3}{x^2 - 1}$$

g)
$$\frac{(x+3)(x-1)}{(x+2)(x+1)}$$

h)
$$\frac{x^3 + 2x^2 - 5x - 6}{x^2 - 4x + 3}$$

ANSWERS

a) $1 - \frac{4}{x+2} + \frac{2}{x-1}$

b) $x+1 + \frac{1}{x-3} - \frac{1}{x+1}$

c) $x + \frac{3}{x-3} + \frac{2}{x-2}$

d) $1 + \frac{1}{x-2} - \frac{1}{x+2}$

e) $x+3 + \frac{2}{x} - \frac{6}{x+1}$

f) $1 + \frac{2}{x-1} - \frac{2}{x+1}$

g) $1 + \frac{3}{x+2} - \frac{4}{x+1}$

h) $x+6 + \frac{12}{x-3} + \frac{4}{x-1}$

Mixed Examples

1.
$$\frac{x}{(x-2)(x-1)}$$

2.
$$\frac{2x+5}{(2x+1)(x+3)}$$

3.
$$\frac{3x+13}{(5x-1)(7x+2)}$$

4.
$$\frac{2x-3}{(x-1)(x-2)(x+3)}$$

5.
$$\frac{8-x}{(x-2)^2(x+1)}$$

6.
$$\frac{x^2-2x+3}{3x^3+x^2}$$

7.
$$\frac{x^2-3}{(x+2)(2x^2+1)}$$

8.
$$\frac{1}{x^4-1}$$

9.
$$\frac{7x+5}{(x-3)(x^2+4)}$$

10.
$$\frac{x^2+15}{(x-1)(x^2+2x+5)}$$

11.
$$\frac{2x^3+5x^2-2x-1}{x^3-x}$$

12.
$$\frac{x^2+9x+5}{(x+1)^3(x-2)}$$

13.
$$\frac{2x^3-17}{(x+1)(x^2+2x+2)}$$

14.
$$\frac{4x^4+5}{(x-1)(x^2-x+3)^2}$$

ANSWERS

1. $\frac{2}{(x-2)} - \frac{1}{(x-1)}$

2. $\frac{1}{5(x+3)} + \frac{8}{5(2x+1)}$

3. $\frac{4}{5x-1} - \frac{5}{7x+2}$

4. $\frac{1}{4(x-1)} + \frac{1}{5(x-2)} - \frac{9}{20(x+3)}$

5. $\frac{1}{x+1} - \frac{1}{x-2} + \frac{2}{(x-2)^2}$

6. $\frac{3}{x^2} - \frac{11}{x} + \frac{34}{3x+1}$

7. $\frac{1}{9(x+2)} + \frac{7(x-2)}{9(2x^2+1)}$

8. $\frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x^2+1)}$

9. $\frac{2}{x-3} - \frac{2x-1}{x^2+4}$

10. $\frac{2}{x-1} - \frac{x+5}{x^2+2x+5}$

11. $2 + \frac{1}{x} + \frac{2}{x-1} + \frac{2}{x+1}$

12. $\frac{1}{x-2} - \frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}$

13. $2 - \frac{19}{x+1} + \frac{13x+17}{x^2+2x+2}$

14. $\frac{1}{(x-1)} + \frac{3x+8}{x^2-x+3} - \frac{3x+20}{(x^2-x+3)^2}$